Averaged null energy condition in curved space

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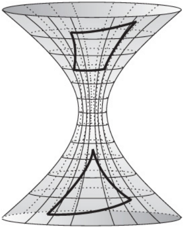

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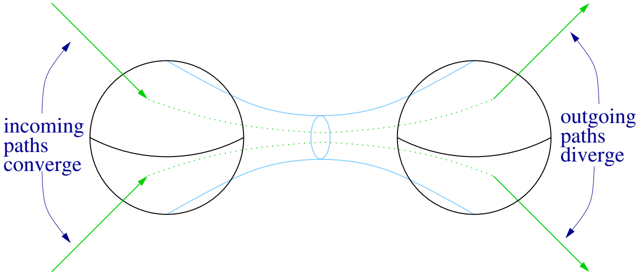

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$\Rightarrow$ Achronal ANEC was proved (Fewster, Olum, Pfenning. 2007) to hold for geodesics in curved space, providing that any curvature stays some minimum distance from the geodesic, which then are travelling in flat space. Here we will try to prove it for geodesics travelling in curved space.

Assumptions

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2. **Coordinate system**: Fermi-like coordinates
3. **Curvature**: We require that $|R_{\mu\nu\rho\sigma}| < R_{max}$ everywhere in M'. We also require the null convergence condition $R_{ab}l^a l^b \geq 0$ for any null vector l which holds whenever the curvature is generated by a classical background whose stress tensor obeys the Null Energy Condition (NEC)

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5. **Quantum field theory**: We consider a quantum scalar field in M, which inside M' is free and minimally coupled.

Quantum inequality

In flat spacetime it was proved (Fewster, Roman 2002) that

$$\int_{-\tau_0}^{\tau_0} d\tau\, T_{ab}(w(\tau)) l^a l^b f(\tau/\tau_0)^2 \geq -\frac{(k_a l^a)^2}{12\pi^2\tau_0^4} \int_{-\tau_0}^{\tau_0} d\tau f''(\tau/\tau_0)^2$$

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where f is compact function with $\int_{-1}^{1} dx f(x)^2 = 1$. Suppose we want to test quantum inequality in a laboratory on the surface of the earth... This is not flat space but it has curvature of order $GM_\oplus/R_\oplus^3$.
$\Rightarrow$ We expect QI to hold with a small correction in globally hyperbolic spacetimes with small curvature: $|R_{abcd}|\tau_0^2 < \epsilon$, where $\epsilon \ll 1$ and small proper acceleration of the timelike paths

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Conjecture

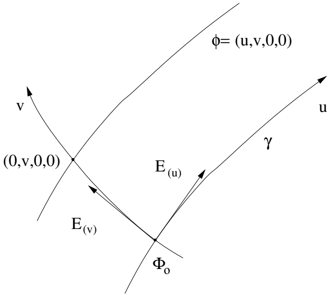
$$\int_{-\tau_0}^{\tau_0} d\tau\, T_{ab}(w(\tau)) l^a l^b f(\tau/\tau_0)^2 \geq -\frac{(k_a l^a)^2}{12\pi^2\tau_0^4} \int_{-\tau_0}^{\tau_0} d\tau f''(\tau/\tau_0)^2 [1 + c\epsilon]$$

Proof of achronal ANEC

1 The parallelogram

Consider the points $\Phi(u,v) = (u,v,0,0)$, null geodesics in M'. The ANEC integral can be writen as

$$A(v) = \int_{-\infty}^{\infty} du T_{uu}(\Phi(u,v))$$



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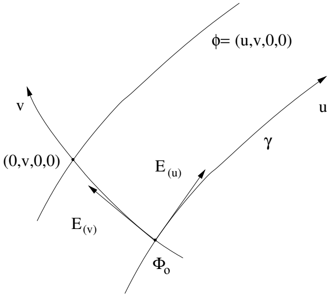

Define $\tau_0 = \gamma^{-\alpha} r$ where $0 < \alpha < 1/3$ and r a positive number with dimensions of length. As $V \to 1$, $\gamma \to \infty$ and $\tau_0 \to 0$. Now consider the points $\Phi_V(\eta, \tau) = \Phi(u, v)$ we can write the ANEC integral as

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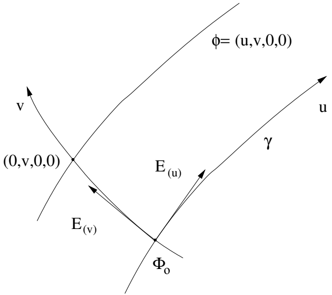

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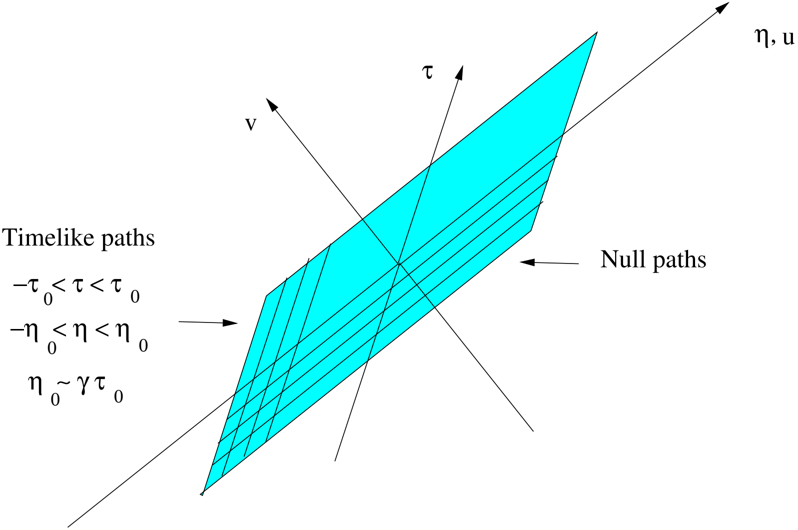
η, u
τ
v
Timelike paths
−τ 0 < τ < τ 0
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η 0 ~ γ τ 0
Null paths

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$$|R_{a'b'c'd'}| < R_{max}$$

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- In our case we can easily prove that $h_{\alpha'\beta'}=O(R_{max}\tau_0^2)$ so $g_{\alpha'\beta'}k^{\alpha'}k^{\beta'}=-1+O(R_{max}\tau_0^2)$ so timelike for sufficiently large γ

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- Using the same arguments we can prove that the acceleration is $|\alpha^{\beta'}|\tau_0 = O(R_{max}\tau_0^2)$
- We apply the QNEI in the globally hyperbolic causal diamond $N = J^+(p) \cap J^-(q)$ which we proved that is inside the tube after the boost

3 Quantum Inequality

We showed that the curvature is bounded, paths Φ_V are timelike and proper acceleration is small. So now we can apply the quantum inequality for small curvature:

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau \, T_{uu}(\Phi_V(\eta,\tau)) f(\tau/\tau_0)^2 \geq -\frac{F\eta_0}{12\pi^2\gamma^2\tau_0^3}[1 + O(R_{max}\tau_0^2)]$$

Where we used $(l_\alpha k^\alpha)^2 \sim 1/\gamma^2$ and $F = \tau_0^{-1} \int d\tau f''(\tau/\tau_0)^2$.

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$\Rightarrow$ Since $\alpha < 1/3$ the lower bound in the first equation goes to zero faster than the upper bound in the second equation. This contradiction proves the theorem.

Conclusions

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