Inadequacies of the Fisher Matrix in Gravitational-wave Parameter Estimation

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The Problem

- Advanced LIGO receives a signal:
  - Past the initial detection, all science potential comes from measurement of system parameters
  - Our parameter estimation accuracy is key to Advanced LIGO/VIRGO's science potential
- Most studies have used the Fisher Information Matrix (the Cramer-Rao bound) to describe our parameter estimation capabilities
- Do our real parameter estimation tools agree with the Fisher Matrix?

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3. Repeat the calculation for different parameters, producing posterior probability distribution via Bayes Theorem

$$p(\theta|s) \propto p(s|\theta)p(\theta)$$

Parameter Estimation (MCMC)

- Still need to effectively sample (15 dimensional!) parameter space
- Use Markov-Chain Monte Carlo:

    1. propose a jump in parameter space
    2. accept if signal fit is better
    3. reject if fit is worse (sometimes)

- "Chains" trace out parameter space efficiently, finding multiple modes despite parameter degeneracies
- Difficult and computationally expensive

Fisher Information Matrix

- Assume gaussian, stationary noise and high signal-to-noise ratio, and define an inner product:

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- To first order,

$$p(\theta|s) \propto p(\theta) \exp\left[-\frac{1}{2}\Gamma_{ab}\Delta\theta^a\Delta\theta^b\right]$$

- where $\Gamma_{ab} \equiv \left\langle \frac{\partial h}{\partial \theta^a} \middle| \frac{\partial h}{\partial \theta^b} \right\rangle$ and $\Sigma^{ab} = (\Gamma^{-1})^{ab}$

(Fisher Matrix)

Fisher Information Matrix Pitfalls

- No (good) way to include prior information
- No global exploration of the parameter space
  - (Multiple peaks)
  - Walls in parameter space
- Highly susceptible to large, unphysical cross correlations
  - Parameters with physical compactification (e.g. angles)
- How high an SNR is required for the linear-signal approximation to be valid?

MCMC vs Fisher Matrix

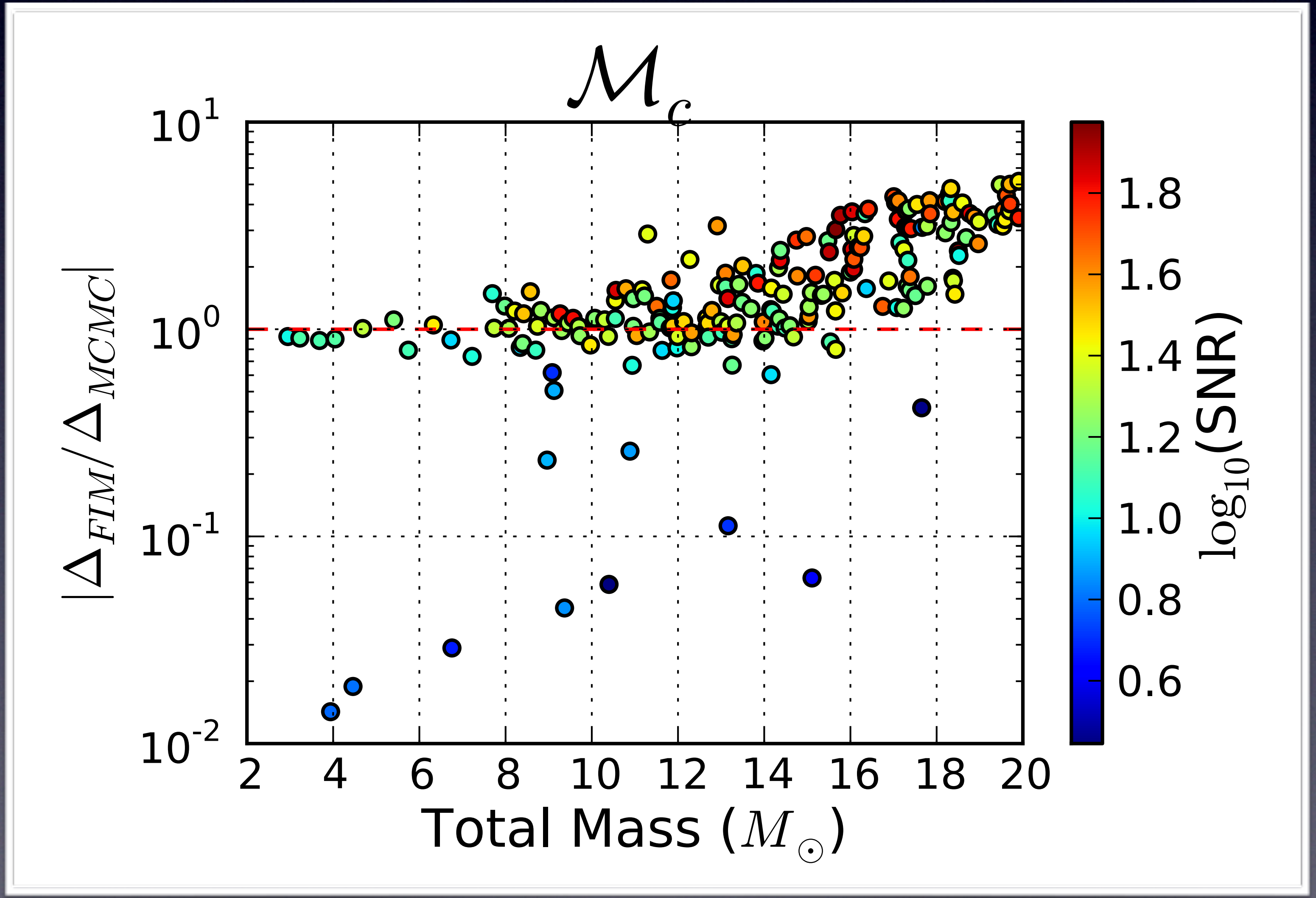
- Used set of 200 non-spinning frequency domain signals
  - TaylorF2 Approximant, Initial LIGO noise
- Compare the MCMC standard deviations to the Fisher Matrix bounds
- Since we assume posterior from MCMC to be correct, normalize Fisher matrix errors by MCMC standard deviations

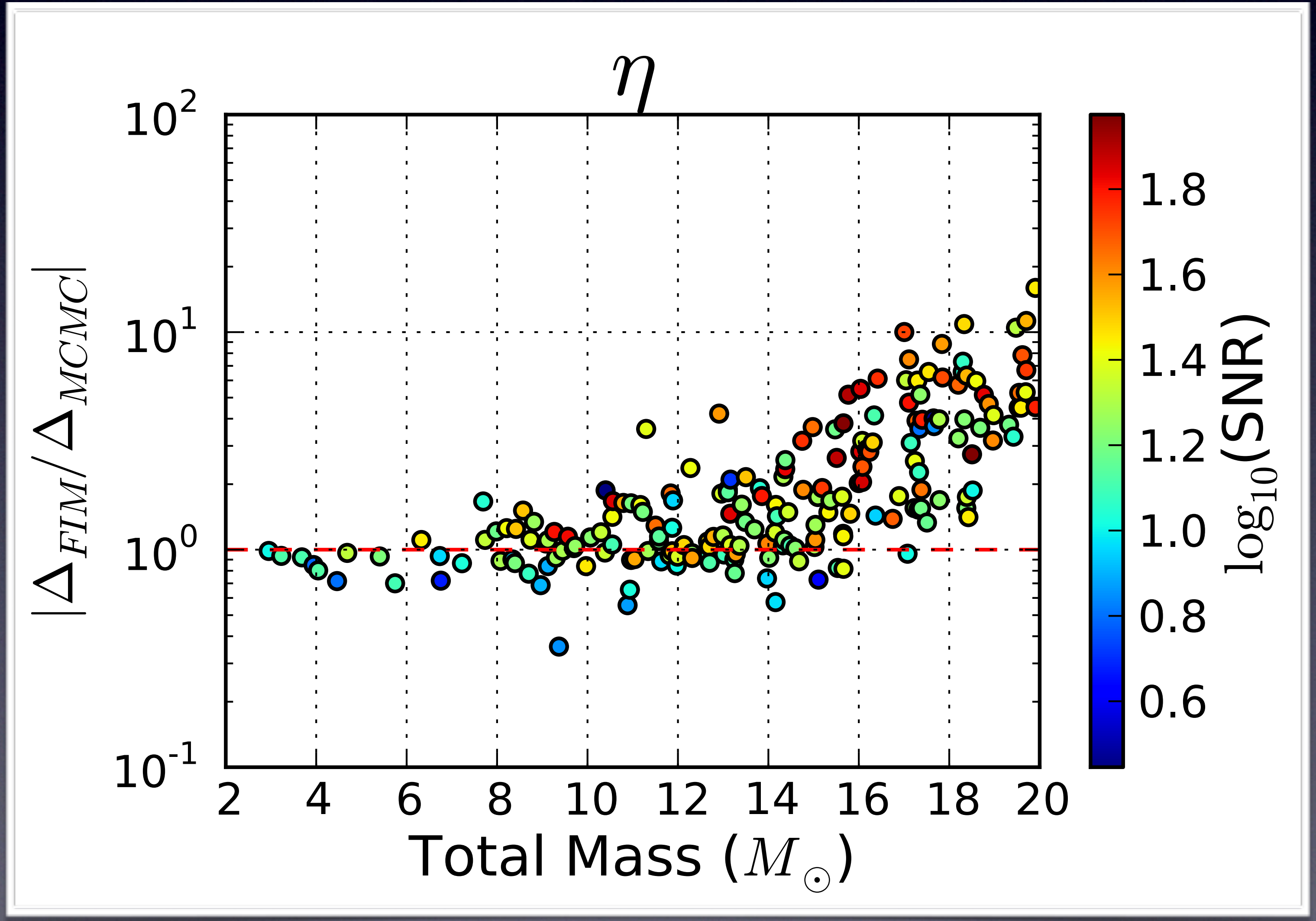
$$\frac{\Delta_{FIM}}{\Delta_{MCMC}}$$

- As Fisher Matrix represents statistical lower bound, we expect the fractional standard deviations to be < 1

MCMC vs Fisher Matrix

- Look at the fractional errors as a function of total mass:



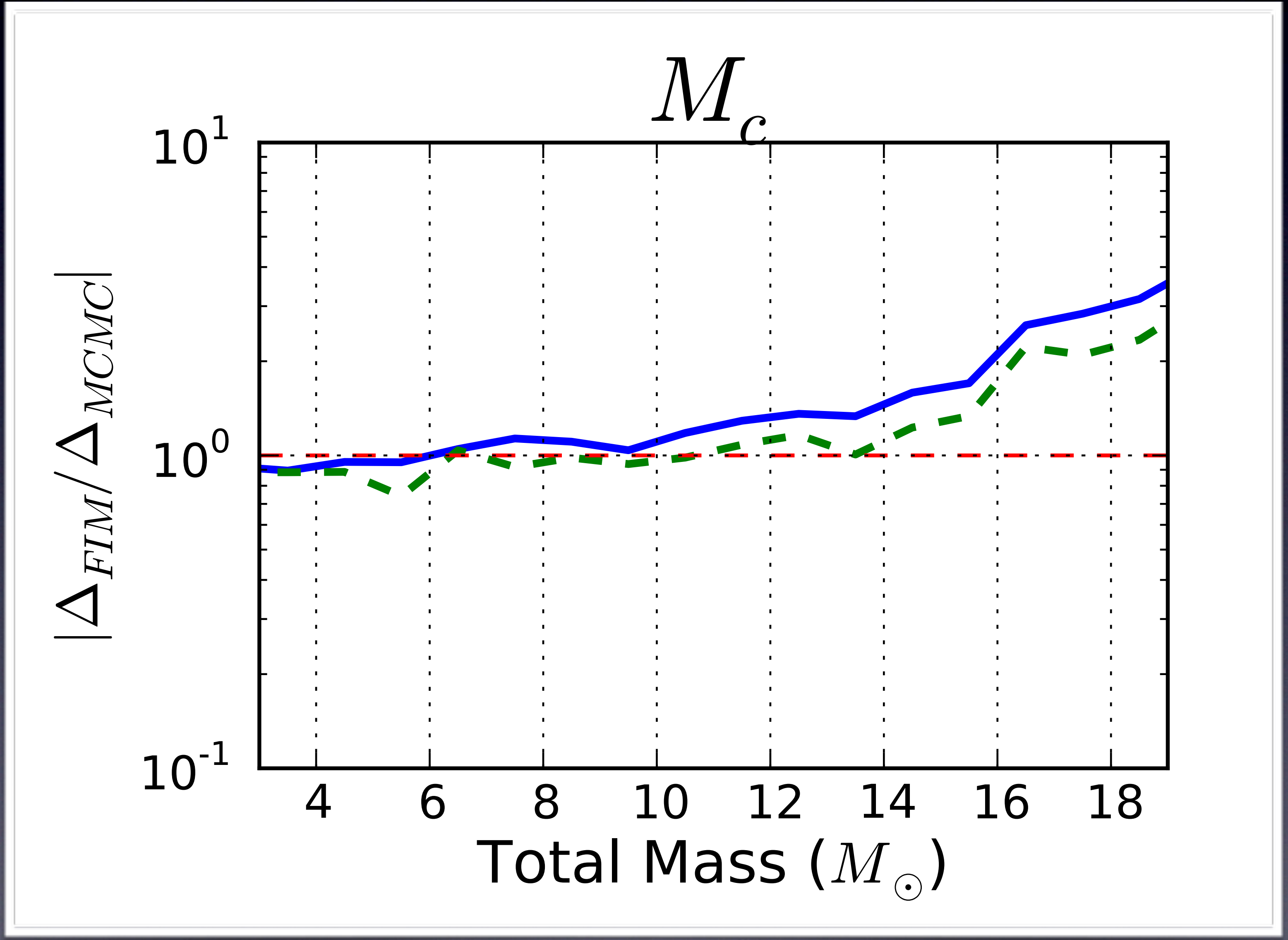


MCMC vs Fisher Matrix

- Instead of undercutting the MCMC errors, the Fisher matrix lower bound is failing,
  - Cramer-Rao Bound is failing, by up to an order-of-magnitude
  - Beginning at $\sim 10 M_\odot$
  - Even for chirp mass, one of the best recovered parameters
  - Combination of two effects: compactified parameter correlations and break-down of Linear Signal Approximation

Compact Parameter Correlations

- No (good) way to include deal with compactified parameters
    - But you can kludge it
- Using priors, you can shrink the overall error ellipsoid volume
- Need global understanding of parameter space



Linear Signal Breakdown

- Fisher matrix still appears to break down past ~$10M_{\odot}$
  - Suggests complete breakdown of gaussian structure of the posterior likelihood space
  - Can test using technique from Valisneri 2008
    - Determines if the 1-Sigma surfaces produced by Fisher matrix are sufficiently narrow for linear-signal approximation to be valid
- Preliminary results indicate that higher-massed systems require significantly higher SNRs for Fisher Matrix to be self-consistent

Bottom Line

- Studies done with Fisher matrix were good first step, but not realistic when compared to proper parameter estimation
    - Can over-estimate the error on compactified parameters, causing cause you to wildly overestimate the errors on other parameters due to correlations
    - The linear-signal approximation easily breaks down, especially at total masses higher than $\sim 10 M_\odot$
    - In order to know where this happens, you need to already know about the parameter space (at which point you might as well do MCMC anyway)