Topological effects in linear gauge theories

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Joint work (in preparation) with Claudio Dappiaggi (Pavia) and Thomas-Paul Hack (Hamburg).

Introduction and motivation

Recently, Dappiaggi et al. considered the free (quantum) vector potential of electromagnetism in curved spacetimes, in the light of general covariance. Their conclusions:

- The Poisson bracket may be degenerate, depending on the topology of the background spacetime.
- The theory shows non-local effects: the degenerate observables may vanish, after embedding a spacetime into a larger one.

The interpretation of these degenerate observables remained to be clarified.

Introduction and motivation

In this talk we use the vector potential to illustrate a general formalism that helps to clarify these issues. We address the following questions:

- How do we compute the degeneracies of the Poisson bracket?
- How do we interpret these degeneracies?

For the vector potential this leads to an apparently new insight in the relation between the Aharonov-Bohm effect and Gauss' law.

Electromagnetism in Minkowski spacetime

In Minkowski spacetime M_0, electromagnetism is described by:

- A Maxwell field $F \in \Omega^2(M_0)$ such that

$$dF = \nabla_{[\mu} F_{\nu\rho]} = 0 \qquad \delta F = \nabla^\mu F_{\mu\nu} = J = 0,$$

- or a vector potential $A \in \Omega^1(M_0)$ such that ($F = dA$)

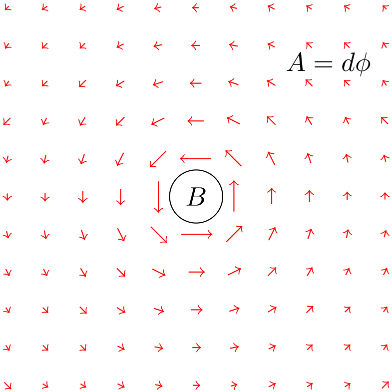
$$\delta dA = J = 0 \qquad A \sim 0 \Leftrightarrow dA = 0,$$

- or a vector potential $A \in \Omega^1(M_0)$ such that

$$\delta dA = J = 0 \qquad A \sim 0 \Leftrightarrow A = d\chi,\ \chi \in \Omega^0(M_0).$$

In general spacetimes these formulations are no longer equivalent!

The Aharonov-Bohm effect allows us to distinguish between A_1 and A_2 when $A_1 - A_2$ is not exact:



$A = d\phi$, in a region outside a coil, is closed, but not exact. Using quantum particles one can measure a phase shift $\sim \oint A$.

We therefore consider the theory for $A \in \Omega^1(M)$ such that

$$\delta dA = 0 \qquad A \sim 0 \Leftrightarrow A = d\chi,\ \chi \in \Omega^0(M).$$

Local observables

If $\Sigma \subset M$ is a Cauchy surface, the space of field configurations is parametrised by initial data:

$$\mathcal{F} = \left\{ E \in \Omega^1(\Sigma) |\ \delta E = 0 \right\} \oplus \left\{ a \in \Omega^1(\Sigma) \right\} / d\Omega^0(\Sigma).$$

A local, linear observable is given by

$$\langle (\alpha, \epsilon), (E, a) \rangle := \int_\Sigma \epsilon \wedge *a - \alpha \wedge *E = \int_\Sigma \epsilon_\mu a^\mu - \alpha_\mu E^\mu,$$

with (α, ϵ) in the dual space

$$\mathcal{F}' = \left\{ \alpha \in \Omega^1_0(\Sigma) \right\} / d\Omega^0_0(\Sigma) \oplus \left\{ \epsilon \in \Omega^1_0(\Sigma) |\ \delta\epsilon = 0 \right\}.$$

The pairing $\langle\ ,\ \rangle : \mathcal{F}' \times \mathcal{F} \to \mathbb{C}$ is non-degenerate in both entries.

Peierls' Poisson bracket

The Poisson bracket on $\mathcal{F}'$ can be obtained from the Lagrangian of the theory by a general procedure due to Peierls (1952). It yields:

$$\{(\alpha_1, \epsilon_1), (\alpha_2, \epsilon_2)\} = \int_\Sigma \epsilon_1 \wedge *\alpha_2 - \alpha_1 \wedge *\epsilon_2.$$

Remarks:

- The Poisson bracket is an important structure e.g. for canonical quantisation (or deformation quantisation).
- The Poisson bracket is an anti-symmetric linear map on (linear) observables in $\mathcal{F}'$. (The symplectic form, on the other hand, is a map on $\mathcal{F}$.)

Degeneracies of the Poisson structure

The Poisson bracket is (in general) degenerate:

$$\{(\alpha,\epsilon),(\alpha',\epsilon')\} = 0 \quad \forall(\alpha',\epsilon')\in\mathcal{F}'$$

$$\Leftrightarrow$$

$$\epsilon = 0,\ \alpha\in\deg(\Sigma) := \left(\Omega^1_0(\Sigma)\cap d\Omega^0(\Sigma)\right)/d\Omega^0_0(\Sigma).$$

I.e. $\alpha = d\beta$, α has compact support, but β does not.

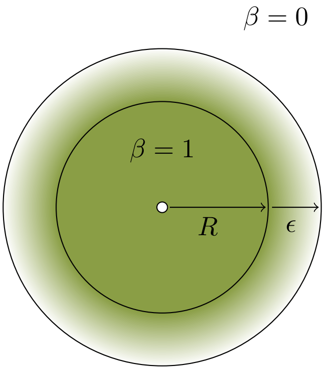
Question:
What do the degenerate observables measure? The Aharonov-Bohm effect?

An example!

Consider an ultrastatic spacetime with $\Sigma := \mathbb{R}^3 \setminus \{0\} = \mathbb{R}_{>0} \times S^2$. Let $\beta \in \Omega^0(\Sigma)$ be

- rotation invariant,
- $\equiv 1$ on $r \leq R$,
- $\equiv 0$ on $r \geq R + \epsilon$,

where r is a radial coordinate. Then $\alpha := d\beta = \beta'(r)dr \ \in \deg(\Sigma)$.



The observable $(\alpha, 0)$ measures (a multiple of) the electromagnetic flux through the shell $1 \leq r \leq 2$.

Gauss' law

All degenerate observables are of this type: they use Gauss' law to measure electric charges which lie outside the spacetime itself.

The (possible) electric charges of a spacetime are characterised by the possible degenerate observables, i.e. by

$$\mathrm{deg}(\Sigma) = \left(\Omega^1_0(\Sigma) \cap d\Omega^0(\Sigma)\right) / d\Omega^0_0(\Sigma).$$

When Σ is compact, $\mathrm{deg}(\Sigma) = \{0\}$.
When $H^1(\Sigma) = \{0\}$, $\mathrm{deg}(\Sigma) = H^1_0(\Sigma)$. A basis of degenerate observables is then indexed by non-contractible spheres in Σ, up to homology.

Electric monopoles

In general, a basis of degenerate observables is indexed by non-contractible spheres in Σ, up to homology, which cut Σ into two non-compact pieces.

A pedagogical example is $\Sigma := S^1 \times S^2$.
Here $H^1_0(\Sigma) \simeq \mathbb{R}$, but $\mathrm{deg}(\Sigma) = \{0\}$ as Σ is compact.

Physical intuition:
Removing any non-contractible sphere from Σ leaves a single connected set. The sphere does not separate a point charge from infinity. There is no charge.

Conclusions

- The Aharonov-Bohm effect motivated the choice of gauge equivalence.
- By general procedures we found the Poisson structure and its degeneracies.
- The degeneracies correspond to Gauss' law and yield a topological formula for electric monopoles.
- The same mathematical argument works for p-form fields and magnetic monopoles, also when source currents are present.
- The same argument should apply to other linearised gauge theories (e.g. linearised GR).

Spacetime formulae

The space of field configurations is

$$\mathcal{F} := \left\{ A \in \Omega^1(M) \middle| \, \delta dA = 0 \right\} / d\Omega^0(M).$$

A local, linear observable is

$$f(A) := \langle f, A \rangle := \int_M f \wedge *A, \quad f \in \Omega^1_0(M).$$

The space of such observables is

$$\mathcal{F}' := \left\{ f \in \Omega^1_0(M) \middle| \, \delta f = 0 \right\} / \delta d\Omega^1_0(M)$$

so that the pairing

$$\mathcal{F}' \times \mathcal{F} \ni (f, A) \mapsto \langle f, A \rangle$$

is non-degenerate in both entries.

Spacetime formulae

The Poisson bracket is

$$\{f_1, f_2\} = \int_M f_1 E f_2,$$

where E is the advanced-minus-retarded fundamental solution of a hyperbolic (Laplace-Beltrami) operator obtained by fixing a Lorenz gauge.

The space of degenerate observables is

$$\{f, f'\} = 0 \quad \forall f' \in \mathcal{F}' \qquad \Leftrightarrow \qquad f \in \Omega^1_0(M) \cap \delta d \Omega^1_{tc}(M),$$

where *tc* means time-like compact support.

Poisson brackets vs. symplectic forms

We may view $\mathcal{F}$ as an infinite dimensional manifold. Then,

$$T\mathcal{F} \simeq \mathcal{F} \times \mathcal{F}, \quad T^*\mathcal{F} \simeq \mathcal{F} \times \mathcal{F}'.$$

The Poisson bracket is a two-vector field P:

$$\{f_1, f_2\} = P^{ab}(f_1)_a(f_2)_b, \quad f_1, f_2 \in T^*_A\mathcal{F} \simeq \mathcal{F}'.$$

One may also consider a symplectic form (up to technicalities)

$$\Omega(\delta_1 A, \delta_2 A) = \Omega_{ab}(\delta_1 A)^a(\delta_1 A)^b, \quad \delta_1 A, \delta_2 A \in T_A\mathcal{F} \simeq F.$$

(See e.g. Lee and Wald (1990).)

In finite dimensions and without degeneracies, P^{ab} and Ω_{ab} are each other's inverses. In general, the situation is not so clear.