# Dynamical and Thermodynamic Stability of Perfect Fluid Stars

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# Motivation

• Hollands & Wald (2012): In vacuum GR, dynamical stability of black holes with respect to axisymmetric perturbations is equivalent to positivity of the canonical energy $\mathcal{E}$ on the space of perturbations that keep $M$ and $J$ fixed to first order. Additionally,

$$\mathcal{E} = \delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \Omega\delta^2 J,$$

which establishes the equivalence between dynamical and thermodynamic stability.

# Motivation

• Can we similarly establish a dynamical stability criterion for perfect fluid stars? Are dynamical and thermodynamic stability equivalent?

• Issue: $\mathcal{E}$ is constructed starting from a Lagrangian, but for perfect fluids one must introduce nonphysical potential fields to have a Lagrangian formulation. These potentials will not in general be stationary even when the physical fields are, and they introduce extra freedom (in addition to gauge).

# Outline

- Overview of the "Lagrangian displacement" formulation of perfect fluids and the definition of $\mathcal{E}$. (Friedman (1978) used $\mathcal{E}$ to establish dynamical instability with respect to non-axisymmetric perturbations.)
- Main results:
  - We establish a criterion for dynamical stability with respect to axisymmetric perturbations which keep the particle number, entropy, and angular momentum of each fluid element fixed – namely $\mathcal{E} \geq 0$ for all such perturbations.
  - We show that for configurations in thermodynamic equilibrium (rigid rotation, uniform redshifted temperature and chemical potential), this condition for dynamical stability is equivalent to the condition for thermodynamic stability.

# Description of the Einstein-Perfect Fluid System

Consider a single component perfect fluid.

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$$

- Can describe the system by the fields:

$$(g_{ab}, n, s, u^a) ,$$

  with all other fluid quantities determined by an equation of state $\rho(n, s)$ and the first law $d\rho = nTds + \left(\frac{\rho+p}{n}\right) dn$.

- Choose to replace $(n, u^a)$ by $N_{abc} \equiv n u^d \epsilon_{abcd}$. Then the system is described by the "physical" fields:

$$(N_{abc}, s, g_{ab}) .$$

# Equilibrium

- Dynamical equilibrium $\equiv$ stationary axisymmetric: $u^a \propto v^a \equiv t^a + \Omega\varphi^a$

- Iyer (1997): $\boxed{\delta M = \int_\Sigma \left(\widetilde{T}\delta\boldsymbol{S} - \widetilde{\mu}\delta\boldsymbol{N} + \Omega\delta\boldsymbol{J}\right)}$,

where $\boldsymbol{S} \equiv s\boldsymbol{N}$, $\boldsymbol{J} \equiv \varphi \cdot T \cdot \boldsymbol{\epsilon}$, $\widetilde{\mu} \equiv |v|\mu$, and $\widetilde{T} \equiv |v|T$.

- Definition: A dynamical equilibrium configuration is in thermodynamic equilibrium iff the total entropy $S = \int_\Sigma \boldsymbol{S}$ is an extremum with respect to perturbations that don't change $M$, the total particle number $N = -\int_\Sigma \boldsymbol{N}$, and the total angular momentum $J = \int_\Sigma \boldsymbol{J}$. Equivalently, iff $\delta M = 0$ for all perturbations that don't change $(N, S, J)$.

$\Longrightarrow$ *A configuration is in thermodynamic equilibrium iff it is a stationary axisymmetric solution that has uniform $\widetilde{T}$, $\widetilde{\mu}$, and $\Omega$ (rigid rotation).*

# Lagrangian Formulation

- There is no unconstrained Lagrangian formulation having $(N_{abc}, s, g_{ab})$ as the varied fields [Schutz & Sorkin (1977)]. One must introduce new "dynamical" fields.

- We use the following Lagrangian formulation:
  - "Fiducial" 4-manifold $F$ with fixed $N^{(f)}_{abc}$ and $s^{(f)}$ satisfying $d\boldsymbol{N}^{(f)} = 0 = \boldsymbol{N}^{(f)} \wedge ds^{(f)}$
  - Dynamical fields: $\phi = \left(g_{ab}, \chi : F \to M\right)$
  - Physical fields: $(N_{abc}, s, g_{ab}) = \left(\chi^* N^{(f)}_{abc}, \chi^* s^{(f)}, g_{ab}\right)$
  - Lagrangian: $\boldsymbol{L} = \boldsymbol{L}^{(g)} + \boldsymbol{L}^{(m)} = \left(\frac{1}{16\pi} R - \rho\right) \boldsymbol{\epsilon}$.

$\Longrightarrow \nabla_a(nu^a) = u^a \nabla_a s = 0.$

# Lagrangian Formulation

• Variational principle: 1-parameter family $\phi(\lambda) = \Big(g_{ab}(\lambda), \chi_\lambda \circ \chi\Big)$.
($\chi : F \to M$ and $\chi_\lambda : M \to M$, with $\chi_0$ the identity).

• The family $\chi_\lambda$ is generated to first order by a vector field $\xi^a$ known as the "Lagrangian displacement".

- First order perturbation given by a pair $\delta\phi = (\xi^a, \delta g_{ab})$.
- Eulerian perturbation: $\delta Q \equiv \frac{d}{d\lambda} Q(\lambda)\big|_{\lambda=0}$.
- Lagrangian perturbation: $\Delta Q \equiv \frac{d}{d\lambda} \chi_{\lambda *} Q(\lambda)\big|_{\lambda=0} = \delta Q + \mathcal{L}_\xi Q$.

• By construction, $\Delta N_{abc} = 0$ and $\Delta s = 0$, which is a physical restriction on the perturbations allowed in this framework.

# The Canonical Energy

- From $\boldsymbol{L}$, construct the symplectic form [Lee & Wald (1990)]:

$$\begin{aligned} \delta \boldsymbol{L} &= \boldsymbol{E} \cdot \delta\phi + d\boldsymbol{\theta}(\phi; \delta\phi) \\ \boldsymbol{\omega}(\phi; \delta_1\phi, \delta_2\phi) &= \delta_2\boldsymbol{\theta}(\phi, \delta_1\phi) - \delta_1\boldsymbol{\theta}(\phi, \delta_2\phi) - \boldsymbol{\theta}(\phi, [\delta_2, \delta_1]\phi) \\ W[\phi; \delta_1\phi, \delta_2\phi] &= \int_\Sigma \boldsymbol{\omega}(\phi; \delta_1\phi, \delta_2\phi) \\ &= \int_\Sigma \left( \frac{1}{16\pi} \delta_1\pi^{ab}\delta_2 h_{ab} + (P_1)_a(\xi_2)^a \right) - (1 \leftrightarrow 2) \end{aligned}$$

- On solutions, $W$ is independent of the choice of $\Sigma$.
- For a stationary background, define

$$\boxed{\mathcal{E}(\delta_1\phi, \delta_2\phi) \equiv W(\phi; \mathcal{L}_t\delta_1\phi, \delta_2\phi)}$$

$\rightarrow$ conserved symmetric bilinear form on perturbations.

# Dynamical Stability

• What subspace $\mathcal{T}$ of axisymmetric perturbations should we investigate stability on? (In the black hole case $\mathcal{T}$ was defined by $\delta M = \delta J = 0$).

• We definitely need to restrict to perturbations which don't change $(N, S, J)$, or else we could not possibly have positivity of $\mathcal{E}$ as a stability criterion.

• We already have the restriction $\Delta \boldsymbol{N} = \Delta \boldsymbol{S} = 0$. (This fixes the particle number and entropy in each fluid element, and therefore also the total $N$ and $S$.)

• We must also somehow restrict the angular momentum perturbation. It seems sensible to restrict to $\Delta \boldsymbol{J} = 0$.

# Dynamical Stability

$\longrightarrow$ $\boxed{\text{Define } \mathcal{T} \text{ by } \Delta \boldsymbol{N} = \Delta \boldsymbol{S} = \Delta \boldsymbol{J} = 0.}$

This is a good choice! To have a stability criterion, it is essential that $\mathcal{E}$ be nondegenerate on $\mathcal{T}$ modulo the perturbations that "don't do anything":

- Pure gauge – these are degeneracies of $W$, and therefore of $\mathcal{E}$.
- "Trivials" – These are defined by $\delta Q = 0$. There are two types in the axisymmetric case:
  - (i) $(\xi, \delta g) = (fu, 0)$. These are the remaining degeneracies of $W$ in the Lagrangian formalism.
  - (ii) $(\xi, \delta g) = (f\varphi, 0)$ for stationary axisymmetric $f$. The perturbations that are symplectically orthogonal to these are exactly the ones that have $\Delta \boldsymbol{J} = 0$.

# The Canonical Energy and Dynamical Stability

Stability criterion for a stationary axisymmetric configuration:

> *Let $\mathcal{T}$ be the set of axisymmetric perturbations that have $\Delta \boldsymbol{N} = \Delta s = \Delta \boldsymbol{J} = 0$. If $\mathcal{E}(\delta\phi, \delta\phi) \geq 0$ for all $\delta\phi \in \mathcal{T}$, then the configuration is stable with respect to all perturbations in $\mathcal{T}$. If $\mathcal{E}$ can be made negative, there is an instability.*

# Thermodynamic Stability

• Main result: *For a background in thermodynamic equilibrium, for any axisymmetric perturbation $\delta\phi$ satisfying $\Delta \boldsymbol{N} = \Delta \boldsymbol{S} = \Delta \boldsymbol{J} = 0$,*

$$\boxed{\mathcal{E}(\delta\phi, \delta\phi) = \delta^2 M - \widetilde{\mu}\delta^2 N - \widetilde{T}\delta^2 S - \Omega\delta^2 J}.$$

• Important ingredient for proof: For any perturbation in the Lagrangian framework, we can write
$(\xi, \delta g) = (\xi, -\mathcal{L}_\xi g) + (0, \Delta g) =$ (pure gauge) $+$ (pure metric).

• $\mathcal{E} > 0$ is the condition for dynamical stability, and the condition for thermodynamic stability is the positivity of the right hand side. Hence, with respect to our considered class of perturbations, dynamical stability is equivalent to thermodynamic stability.

$\rightarrow$ Future: Is this result true for a more general class of perturbations, i.e., those which just fix the *total* $N$, $S$, and $J$ to first order?

Thank You!