Negative Canonical Energy and Exponential Growth

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work in progress with S. Hollands and K. Prabhu

Stability and Canonical Energy

In recent work, S. Hollands and I showed that the necessary and sufficient condition for linear stability of a black hole with respect to axisymmetric perturbations is positivity of canonical energy

$$\mathcal{E} = \int_\Sigma \omega(g; \gamma, \pounds_t \gamma)$$

on the subspace of perturbations γ with $\delta M = \delta J = 0$. We also showed that

$$\mathcal{E} = \delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_A \Omega_A \delta^2 J_A$$

thus showing that dynamical stability is equivalent to thermodynamic stability on this subspace of perturbations.

The argument for stability when $\mathcal{E} \geq 0$ involved its non-degeneracy, thereby providing a positive definite norm on perturbations. The argument for instability of a perturbation with $\mathcal{E} < 0$ paralleled an argument previously given by Friedman and Schutz and Friedman: If the perturbation were to decay (or "settle down" to a stationary final state), the canonical energy would decay to zero, but we proved that the net flux of canonical energy both to infiniity and through the horizon must be positive. Thus, the perturbation cannot decay and presumably must grow.

Exponential Growth: Scalar Field Analog

One would be natural to expect that perturbations with $\mathcal{E} < 0$ should contain exponentially growing modes. For a scalar field, this can be proven as follows:

Write the equations of motion in first order form

$$\dot{\phi} = Kp$$

$$\dot{p} = -U\phi$$

where $Kp = Np$ and $U\phi = -D^a(ND_a\phi) + NV(x)$, where N is the lapse. The analog of canonical energy is

$$E = (p, Kp)_{L^2} + (\phi, U\phi)_{L^2}$$

Write the equations in second order form,

$$\ddot{\phi} = -KU\phi$$

Let $\mathcal{S}$ be the subspace of smooth elements of L^2 that vanish at at the bifurcation surface B and hence are in the range of K. Define a new inner product on $\mathcal{S}$ by

$$(\phi_1, \phi_2) = (\phi_1, K^{-1}\phi_2)_{L^2}$$

Complete $\mathcal{S}$ in this inner product to obtain a Hilbert space $\mathcal{H}$. The operator $A \equiv KU$ is a symmetric operator on $\mathcal{H}$ defined on the dense domain $\mathcal{S}$. Extend A to a self-adjoint operator (probably unique), also denoted A. View the equations of motion as an equation for vectors

$|\phi\rangle \in \mathcal{H}$

$$|\ddot{\phi}\rangle = -A|\phi\rangle$$

The general solution is

$$\begin{aligned} \phi(t) &= \cos[A_+^{1/2}t]P_+\phi_0 + \sin[A_+^{1/2}t]A_+^{-1/2}P_+\dot{\phi}_0 \\ &\quad + \cosh[|A_-|^{1/2}t]P_-\phi_0 + \sinh[|A_-|^{1/2}t]A_-^{-1/2}P_-\dot{\phi}_0 \end{aligned}$$

where P_+ and P_- are the projections to the positive and negative parts of A. From this formula, it follows that if $|\phi_0\rangle \in \mathcal{S}$ and $\langle\phi_0|U|\phi_0\rangle < 0$, then the $\mathcal{H}$-norm of the solution generated by $(\phi_0, 0)$ grows exponentially with time.

Gravitational Case

One would like to apply similar arguments to the gravitational case. However, a number of difficulties arise:

- Difficulty: The initial data (q_{ab}, p_{ab}) is not free but must satisfy the (linearized) constraint equations, which are PDEs. Also, certain boundary conditions at the horizon must be satisfied. Thus, one cannot simply work on an L^2 Hilbert space as in the scalar case.

- Solution: Satisfaction (in the "weak" sense) of the constraints and boundary conditions is equivalent to L^2 orthogonality to the symplectic complement of certain gauge transformations. This defines a subspace of the L^2 Hilbert space.

- Difficulty: There is gauge freedom in the gravitational case. The gauge must be completely fixed to obtain a well defined dynamics and to ensure that any exponential growth that may occur is not "pure gauge."
- Solution: The gauge may be fixed completely by L^2 orthogonality to the allowed gauge transformations.

- Difficulty: The initial data q_{ab} and p_{ab} do not satisfy the same boundary conditions at the horizon and thus do not live in the same Hilbert subspace. This causes difficulties in writing the equations of motion in second order form.

- Solution: Work instead with the "transverse traceless" parts of q_{ab} and p_{ab} in a particular decomposition, which do satisfy the same boundary conditions at the horizon.

- Difficulty: The kinetic energy operator

$$Kp_{ab} = 2N(p_{ab} - \frac{1}{2}ph_{ab}) + 2D_{(a}n_{b)}$$

is not manifestly positive definite.

- Solution: K can be proven to be positive definite.

- Difficulty: It is unclear what the range of K is and what lies in the Hilbert space defined by the "K^{-1} inner product."
- Solution(?): We are working on this!