Non-trivial vacua

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\[ \mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b f_{ab}(\phi) \]

- If \( \text{sign} (f_{ab}) = (+ + \cdots +) \) well-defined
  (usually associated with compact space)

- If \( \text{sign} (f_{ab}) = (- + \cdots +) \) \( NOT \) well-defined
  (usually associated with non-compact space)

mode associated with the negative direction is a \textit{ghost}
**Σ-model** (for non-compact groups)

\[ \eta_{\mu\nu} \rightarrow g_{\mu\nu}(x) \]

\[ \mathcal{L}_{\text{Polyakov}} = -\frac{1}{2} \sqrt{-g(x)g^{\mu\nu}(x)} \partial_{\mu} \phi^a \partial_{\nu} \phi^b f_{ab}(\phi) \]

Diff invariance \quad \text{would-be ghost is gauge mode}

In 2d integrating out \( g_{\mu\nu} \) leads to the Nambu-Goto action

\[ \mathcal{L}_{\text{Nambu-Goto}} = -\frac{1}{2} \sqrt{-\det \left( \partial_{\mu} \phi^a \partial_{\nu} \phi^b f_{ab}(\phi) \right)} \]
Nambu-Goto action

\[ \mathcal{L}_{NG} = \det K = \mathcal{E} \mathcal{E} K^d. \]

If K is diagonalizable

with

\[ K^\mu_\nu = \sqrt{\eta^{\mu\alpha}} \partial_\alpha \phi^a \partial_\nu \phi^b f_{ab} (\phi) \]

\[ \det \left( \frac{\delta^2 \mathcal{L}_{NG}}{\delta \dot{\phi}^a \delta \dot{\phi}^b} \right) = 0 \quad \rightarrow \quad \text{would-be ghost is absent} \]
Generalization...

\[ \mathcal{L}_{\text{NG}} = \mathcal{E} \mathcal{E} K^d \quad \longrightarrow \quad \mathcal{L}_n = \mathcal{E} \mathcal{E} K^n \]

with \[ K^\mu_\nu = \sqrt{\eta^{\mu\alpha} \partial_\alpha \phi^a \partial_\nu \phi^b f_{ab}(\phi)} \]

\[ \det \left( \frac{\delta^2 \mathcal{L}_n}{\delta \dot{\phi}^a \delta \dot{\phi}^b} \right) = 0 \quad \longrightarrow \quad \text{would-be ghost is also absent} \]
Generalization...

\[ \mathcal{L}_{\text{NG}} = \varepsilon \varepsilon K^d \rightarrow \mathcal{L}_n = \varepsilon \varepsilon K^n \]

The difference between \( \mathcal{L}_n \) and the NG action is that for \( n \neq 0, d \) the constraint is not associated with a symmetry.

- For \( n = d \) there are \( (D - d) \) dynamical dof.
- For \( n \neq 0, d \) there are \( (D - 1) \) dynamical dof (if \( D \geq d \)).

Field space: \( D \)-dim
Spacetime: \( d \)-dim
\[ D \geq d > 2 \]
Perturbative dofs

- Focus on $\mathcal{L}_2$ in $d = D = 4$ dimensions with $f_{ab} = \eta_{ab}$

\[
\mathcal{L}_2 = -\frac{1}{8} \left( \text{Tr} \left( K^2 \right) - (\text{Tr}K)^2 \right)
\]

This theory has $D - 1 = 3$ dofs
Perturbative dofs

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This theory has $D - 1 = 3$ dofs

- 1 dof is lost on the trivial background $\phi^a = x^\mu \delta^a_\mu$. 

$$K_\nu^\mu = \sqrt{\eta^{\mu\alpha} \partial_\alpha \phi^a \partial_\nu \phi^b \eta_{ab}}$$
Perturbative dofs

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- 1 dof is lost on the trivial background $\phi^a = x^\mu \delta_\mu^a + A^a(x)$

\[ \mathcal{L}_2 = -\frac{1}{4} F_{ab}^2 + O(\partial A)^3 \]

\[ F_{ab} = \partial_a A_b - \partial_b A_a \]

$U(1)$-symmetry $\rightarrow$ only 2 propagating dofs...
Consequences for Gravity

\[ \mathcal{L} = \frac{M_{Pl}^2}{2} \sqrt{-g} \left( R[g] - \frac{1}{4} m^2 \left( \text{Tr} (K^2) - (\text{Tr} K)^2 \right) \right) \]

On the trivial background: \[ \phi^a = x^a + A^a + \partial^a \chi , \quad g_{\mu \nu} = \eta_{\mu \nu} \]

\( \chi \) is a gauge mode...

\( \chi \) only becomes dynamical once interactions with the graviton are considered

\[ g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \]

\[ h^{\mu \nu} (\partial_\mu \partial_\nu \chi - \Box \chi \eta_{\mu \nu}) \]
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At the origin of vDVZ
Need for Vainshtein,
Strong coupling
no exact FRWL solutions,
SL (?)

...
Consequences for Gravity

\[ M^2_{\text{pl}} (\equiv 1 \text{ eV}^{-2}) \]

On the trivial background:

\[ \chi \]

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\[ \chi \]

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At the origin of vDVZ

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…
MG on AdS

\[ \mathcal{L} = \frac{M_{Pl}^2}{2} \sqrt{-g} \left( R[g] - \frac{1}{4} m^2 \left( \text{Tr} \left( K^2 \right) - (\text{Tr}K)^2 \right) \right) \]

On the AdS background:

\[ \phi^a = x^a + A^a + \partial^a \chi \]

\[ g_{\mu \nu} = \gamma_{\mu \nu}^{(\text{AdS})} + h_{\mu \nu} \]

Helicity-0 mode is dynamical on that background

\[ \mathcal{L}_\chi = -\frac{1}{2} \left( \frac{M_{Pl}^2 m^2}{L^2} + M_{Pl}^2 m^4 \right) (\partial \chi)^2 + m^2 \chi T \]
MG on AdS

\[ \mathcal{L} = \frac{M_{\text{Pl}}^2}{2} \sqrt{-g} \left( R[g] - \frac{1}{4} m^2 \left( \text{Tr} \left( K^2 \right) - (\text{Tr} K)^2 \right) \right) \]

\[ L^{-1} \ll m \quad \Rightarrow \quad \hat{\chi} = M_{\text{Pl}} m^2 \chi \]

\[ \mathcal{L}_\chi = -\frac{1}{2} (\partial \hat{\chi})^2 + \frac{1}{M_{\text{Pl}}} \chi T \]

Strong coupling scale:

\[ \Lambda_3 = \left( M_{\text{Pl}} m^2 \right)^{1/3} \]

\[ L^{-1} \gg m \quad \Rightarrow \quad \hat{\chi} = \frac{M_{\text{Pl}} m}{L} \chi \]

\[ \mathcal{L}_\chi = -\frac{1}{2} (\partial \hat{\chi})^2 + \frac{\ll 1}{m L} \frac{m L}{M_{\text{Pl}}} \chi T \]

Strong coupling scale:

\[ \Lambda_* = \left( \frac{M_{\text{Pl}} m}{L} \right)^{1/3} \gg \Lambda_3 \]

AdS curvature makes it much better defined...
MG on AdS

- On AdS there is no linearized vDVZ discontinuity

- Strong coupling scale can be pushed higher

- There may be a local, causal, unitary UV completion of a single massive graviton

- AdS/CMT may be one of massive gravity raison d’être

Porrati, 2001
\[ \Lambda = (M_{Pl} m/L)^{1/3} \]

\[ \Lambda = (M_{Pl} m^2)^{1/3} \]

MG on AdS

MG on Minkowski

GR on AdS + 3 decoupled dof

GR on Mink + 3 decoupled dof

\[ m \to 0 \]

\[ \Lambda \to 0 \]

\[ L^{-1} \to 0 \]

(Vainshtein)
Copping with Minkowski...

- Can we mimic the effect of AdS while remaining on Minkowski?

Work in progress with Shuang-Yong Zhou and Andrew Tolley
Copping with Minkowski...

- Can we mimic the effect of AdS while remaining on Minkowski?

Switch gravity off

\[ \mathcal{L} = \frac{M_{P1}^2}{2} \sqrt{-g} \left( R[g] - \frac{1}{4} m^2 \left( \text{Tr} \left( K^2 \right) - (\text{Tr}K)^2 \right) \right) \]

\[ \phi^a = x^a + A^a + \partial^a \chi, \quad g_{\mu\nu} = \eta_{\mu\nu} \]
Can we mimic the effect of AdS while remaining on Minkowski?

Switch gravity off

$$
\mathcal{L} = \frac{M_{Pl}^2}{2} \sqrt{-g} \left( R[g] - \frac{1}{4} m^2 \left( \text{Tr} \left( K^2 \right) - \left( \text{Tr} K \right)^2 \right) \right)
$$

$$
\phi^a = x^a + A^a + \partial^a \chi, \quad g_{\mu\nu} = \eta_{\mu\nu}
$$
Non-trivial vacuum

\[ \mathcal{L} = -\frac{1}{8} \left( \text{Tr} \left( K^2 \right) - (\text{Tr}K)^2 \right) \]

e.g. Plane waves are exact vacuum solutions

\[ \tilde{\phi}^a = x^a + F(t - x) \delta_y^a \]

DOFs about that background

\[ \phi^a = \tilde{\phi}^a + V^a \]

\[ \mathcal{L} = -M_{ab}^{\mu \nu}(F) \partial_\mu V^a \partial_\nu V^b + \mathcal{O}(\partial V)^3 \]

Breaks the \( U(1) \) symmetry \( \rightarrow \) 3 dof
Consistent vacuum

We can easily find local vacua

$$\bar{\phi}^a = \bar{M}^a_b x^b + \frac{1}{L} \bar{P}^a_{bc} x^b x^c + \ldots$$

On which there are 3 dofs and Hamiltonian is positive definite

Could be a consistent $\Sigma$-model on Minkowski

Without the need for gravity

With a strong coupling scale $\Lambda$, $\Lambda_3 \ll \Lambda \ll L^{-1}$
Consequences for MG

\[ \mathcal{L}_{\text{MG}} = \mathcal{L}_{\text{GR}} - m^2 M_{\text{Pl}}^2 \left( \text{Tr} \left( K^2 \right) - (\text{Tr} K)^2 \right) + \mathcal{L}_{\text{matter}}(g_{\mu\nu}, \psi) \]

By working about a non-trivial vacuum, we can easily take a \( \Lambda_2 \) decoupling limit of MG

\[ \bar{\phi}^a = \bar{M}_b^a x^b + \frac{1}{L} \bar{P}_c^{ab}{x^b}{x^c} + \cdots \]
Consequences for MG

\[ \mathcal{L}_{\text{MG}} = \mathcal{L}_{\text{GR}} - m^2 M_{\text{Pl}}^2 \left( \text{Tr} \left( K^2 \right) - \left( \text{Tr} K \right)^2 \right) + \mathcal{L}_{\text{matter}}(g_{\mu \nu}, \psi) \]

\[ \bar{\phi}^a = \bar{M}^a_b x^b + \frac{1}{L} \bar{P}_{bc} x^b x^c + \cdots \]

By working about a non-trivial vacuum, we can easily take a \( \Lambda_2 \) decoupling limit of MG

\[ M_{\text{Pl}} \to \infty \]

\[ \Lambda_2 = (m M_{\text{Pl}})^{1/2} \to \text{const} \]

\[ \mathcal{L}_{\text{MG}} \to h \partial^2 h + \frac{1}{M_{\text{Pl}}} hT + \Lambda_2^4 \mathcal{L}(\eta_{\mu \nu}, V^a) \]
Partially Massless

- Massive Gravity on de Sitter

\[ \mathcal{L}_\chi = -\frac{M_{\text{Pl}}^2 m^2}{2} \left( m^2 - 2H^2 \right) (\partial \chi)^2 \]

- Helicity-0 mode is non-dynamical if \( m^2 = 2H^2 \) (Higuchi bound 1986)

open the door for a potential PM gravity
(MG with additional symmetry with no helicity-0 mode)

But in MG it was shown that the symmetry does not remain at higher order and the helicity-0 mode reappears making the dS background infinitely strongly coupled
Partially Massless

- Massive Gravity on de Sitter

\[ L_\chi = -\frac{M_{\text{Pl}}^2 m^2}{2} \left( m^2 - 2H^2 \right) (\partial \chi)^2 \]

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Could the conclusions be different in Bi-Gravity???
PM bi-Gravity

$$\mathcal{L}_{\text{BiG–PM}} = \mathcal{L}_{\text{GR}}(g) + \mathcal{L}_{\text{GR}}(f) + m^2 M_{\text{P1}}^2 \mathcal{L}_K(\phi^a)$$

The helicity-0 mode acquires a kinetic term in non-trivial backgrounds:

$$\phi^a = \bar{M}^a_b x^b + \frac{1}{L} \bar{P}^a_{bc} x^b x^c + \cdots$$

$$\mathcal{L}_{\text{BiG–PM}} = \mathcal{L}_{\text{GR}}(g) + \mathcal{L}_{\text{GR}}(f) + \mathcal{L}_F(V^i) + \mathcal{O}(m^2)$$

The theory has 7 dofs on that vacuum. It cannot be enjoying a symmetry (ie just like there is no non-linear PM gravity, there is no non-linear PM bi-gravity)
The potential of massive gravity can be thought of as a $\Sigma$-model for a non-compact group.

There are (local) non-trivial vacua for which all the dofs are healthy and stable.

On these vacua MG can have a much higher strong coupling scale that previously anticipated and no fifth force in the decoupling limit.

This also means that there cannot be a PM theory of neither gravity nor bi-Gravity or multi-Gravity.