A Stückelberg Approach to Quadratic Curvature Gravity

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Higher curvature gravity and effective field theory

Einstein gravity is non-renormalizable ⇒ incomplete or strongly coupled at high energies
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We generally interpret it as a low energy effective field theory

\[ S \sim \frac{1}{M_P^2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{m^2} \left( a_2 R^2 + \cdots \right) + \frac{1}{m^4} \left( a_3 R^3 + \cdots \right) + \cdots \right] \]

Higher derivative terms suppressed by mass scale of new physics
coefficients sensitive to details of new physics

Observables are to be calculated *perturbatively*, order by order in powers of: \( \frac{E}{M_P}, \frac{E}{m} \)
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Observables are to be calculated \textit{perturbatively}, order by order in powers of: \( \frac{E}{M_P}, \frac{E}{m} \)

We do not worry about ghosts/superluminality/cauchy breakdown etc. associated with the higher derivative terms: these are non-perturbative in \( E/m \)
Higher curvature gravity and effective field theory

Nevertheless, there is a long and fine tradition of ignoring the perturbativity requirement and asking what higher curvature terms have to say non-perturbatively

Motivations:

• gain intuition about the gravitational effects higher-scale physics might produce

• display various pathologies that a UV completion must ultimately overcome

• try to find a UV complete theory of gravity

e.g. Stelle (1976 — present)
Quadratic Curvature gravity

Most general action up to fourth order in derivatives:

\[ S = M_P^2 \int d^4 x \sqrt{-g} \left[ \frac{1}{2} R + \frac{1}{12m^2} R^2 + \frac{1}{4M^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right] \]
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Degrees of freedom around Minkowski:

- massless graviton
- massive graviton
- massive scalar

one of these is always ghostly
Quadratic Curvature gravity as massive gravity

The spectrum contains a massive spin-2 $\Rightarrow$ it is a theory of massive gravity
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Massive gravity is usually not renormalizable: generic strong coupling at \( \Lambda_5 \sim (M_P M^4)^{1/5} \)

At best, it is: \( \Lambda_3 \sim (M_P M^2)^{1/3} \) (as in dRGT theory and HR bi-metric theory)
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Longitudinal modes of massive graviton described by a non-renormalizable galileon lagrangian:

$$-3(\partial\hat{\phi})^2 + \frac{6(6c_3 - 1)}{\Lambda_3^3} (\partial\hat{\phi})^2 \Box \phi - 4 \frac{(6c_3 - 1)^2 - 4(8d_5 + c_3)}{\Lambda_3^6} (\partial\hat{\phi})^2 \left([\hat{\Pi}]^2 - [\hat{\Pi}^2]\right)$$

$$- \frac{40(6c_3 - 1)(8d_5 + c_3)}{\Lambda_3^9} (\partial\hat{\phi})^2 \left([\hat{\Pi}]^3 - 3[\hat{\Pi}^2][\hat{\Pi}] + 2[\hat{\Pi}^3]\right) + \text{scalar-tensor mixing}$$
Quadratic Curvature gravity as massive gravity

Quadratic gravity, being renormalizable, should have no strong coupling scale
Quadratic Curvature gravity as massive gravity

Quadratic gravity, being renormalizable, should have *no* strong coupling scale

How does its massive graviton evade the ubiquitous $\Lambda_3$ strong coupling?
Quadratic Curvature gravity as massive gravity

Quadratic gravity, being renormalizable, should have \textit{no} strong coupling scale

How does its massive graviton evade the ubiquitous $\Lambda_3$ strong coupling?

We will use the Stückelberg trick to easily see how

We use techniques developed in: Claudia de Rham, Gregory Gabadadze, David Pirtskhalava, Andrew Tolley, Itay Yavin
“Nonlinear Dynamics of 3D Massive Gravity”
arXiv:1103.1351
Re-write as a second-order theory

Quadratic gravity is a 4-th order theory:

\[ M_P^2 \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + \frac{1}{12m^2} R^2 + \frac{1}{4M^2} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \right] \]

Any non-degenerate higher order theory can be written in second order form using auxiliary variables.
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\]

Any non-degenerate higher order theory can be written in second order form using auxiliary variables.

First eliminate f(R) part by introducing a scalar: \( \phi = R \)

\[
M_P^2 \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \left( 1 + \frac{\phi}{3m^2} \right) R - \frac{1}{12m^2} \phi^2 + \frac{1}{4M^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right]
\]

Go to Einstein frame: \( \phi = 3m^2 (e^\psi - 1) \quad g_{\mu\nu} \rightarrow e^{-\psi} g_{\mu\nu} \)

\[
M_P^2 \int d^4 x \sqrt{-g} \left[ \frac{1}{2} R - \frac{3}{4} (\partial \psi)^2 - \frac{3}{4} m^2 e^{-2\psi} (e^\psi - 1)^2 + \frac{1}{4M^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right]
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Go to Einstein frame: $\phi = 3m^2 (e^\psi - 1)$ $g_{\mu\nu} \rightarrow e^{-\psi} g_{\mu\nu}$

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Next eliminate Weyl part by introducing a tensor: $f_{\mu\nu} = \frac{1}{M^2} (R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu})$

$$M_P^2 \int d^4 x \sqrt{-g} \left[ \frac{1}{2} R - \frac{3}{4} (\partial \psi)^2 - \frac{3}{4} m^2 e^{-2\psi} (e^\psi - 1)^2 + f_{\mu\nu} G_{\mu\nu} - \frac{1}{2} M^2 (f_{\mu\nu} f^\mu^\nu - f^2) \right]$$
**Linear spectrum**

Easiest way to see the spectrum: expand in linear fluctuations about flat space and diagonalize:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

\[ h_{\mu\nu} = 2(h'_{\mu\nu} + f_{\mu\nu}) \]

Action to second order in fluctuations: \( h'_{\mu\nu}, f_{\mu\nu}, \psi \)

\[ M_p^2 \int d^4x \left[ -\frac{3}{4} (\partial \psi)^2 - m^2 \psi^2 + \frac{1}{2} h'''_{\mu\nu} (\mathcal{E} h')_{\mu\nu} - \frac{1}{2} f'''_{\mu\nu} (\mathcal{E} f)_{\mu\nu} - \frac{1}{2} M^2 (f_{\mu\nu} f^{\mu\nu} - f^2) \right] \]
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Action to second order in fluctuations: \( h'_{\mu\nu}, f_{\mu\nu}, \psi \)

\[
M_P^2 \int d^4x \ - \frac{3}{4} ((\partial \psi)^2 - m^2 \psi^2) + \frac{1}{2} h'^{\mu\nu} (\mathcal{E} h')_{\mu\nu} - \frac{1}{2} f^{\mu\nu} (\mathcal{E} f)_{\mu\nu} - \frac{1}{2} M^2 (f_{\mu\nu} f^{\mu\nu} - f^2)
\]

massive scalar field \( \psi \), with mass squared \( m^2 \)

massless spin-2 field \( h'_{\mu\nu} \)

massive (ghost) spin-2 field \( f_{\mu\nu} \), with mass squared \( M^2 \)
Stückelberg-ing

Quadratic curvature gravity is already diff invariant: no obvious broken gauge symmetry for Stückelberg fields to restore
Stückelberg-ing

Quadratic curvature gravity is already diffeomorphism invariant: no obvious broken gauge symmetry for Stückelberg fields to restore

Really a bi-metric theory $\Rightarrow$ introduce Stückelbergs to restore a second diff invariance

$$f_{\mu\nu} \rightarrow f_{\mu\nu} + \nabla_\mu \tilde{V}_\nu + \nabla_\nu \tilde{V}_\mu, \quad \tilde{V}_\mu = V_\mu + \partial_\mu \pi$$

$$M_P^{D-2} \int d^D x \sqrt{-g} \left[ \frac{1}{2} R + f^{\mu\nu} G_{\mu\nu} - \frac{1}{2} M^2 \left( f_{\mu\nu} f^{\mu\nu} - f^2 \right) - \frac{1}{2} M^2 F_{\mu\nu}^2 \right.$$  

$$\left. + 2 M^2 R_{\mu\nu} \tilde{V}^\mu \tilde{V}^\nu - 2 M^2 f^{\mu\nu} \left( \nabla_\mu \tilde{V}_\nu - g_{\mu\nu} \nabla \cdot \tilde{V} \right) \right],$$

second diff invariance:  

$$\delta f_{\mu\nu} = \nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu, \quad \delta V_\mu = -\Lambda_\mu + \partial_\mu \Lambda, \quad \delta \pi = \Lambda$$
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\[ M_P^{D-2} \int d^Dx \sqrt{-g} \left[ \frac{1}{2} R + f^{\mu\nu} G_{\mu\nu} - \frac{1}{2} M^2 f_{\mu\nu} f^{\mu\nu} - f^2 \right] - \frac{1}{2} M^2 F_{\mu\nu}^2 \]

\[ + 2 M^2 R_{\mu\nu} \tilde{V}^\mu \tilde{V}^\nu - 2 M^2 f^{\mu\nu} \left( \nabla_\mu \tilde{V}_\nu - g_{\mu\nu} \nabla \cdot \tilde{V} \right) \]

second diff invariance: \[ \delta f_{\mu\nu} = \nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu, \quad \delta V_\mu = -\Lambda_\mu + \partial_\mu \Lambda, \quad \delta \pi = \Lambda \]

Full non-linear DOF count (D=4):

\[
\begin{align*}
\text{(25 fields)} & \quad - \quad 2 \times (9 \text{ gauge symmetries}) \quad = \quad (7 \text{ degrees of freedom}) \\
\begin{cases}
f_{\mu\nu} & 10 \\
g_{\mu\nu} & 10 \\
\Lambda_\nu & 4 \\
\Lambda & 1 \\
\xi^\mu & 4 \\
\Lambda_\mu & 4 \\
\tilde{V}_\mu & 4 \\
\pi & 1 \\
\end{cases}
\end{align*}
\]

massless graviton 2

massive graviton 5

No Boulware-Deser ghost associated with the massive graviton ⇒ expect a \( \Lambda_3 \) strong coupling scale
Decoupling limit

Canonically normalize:

\((h_{\mu\nu}, f_{\mu\nu}) \sim \frac{1}{M_P^{D-1}}(\hat{h}_{\mu\nu}, \hat{f}_{\mu\nu}), \quad V_{\mu} \sim \frac{1}{M_P^{D-1}M} \hat{V}_{\mu}, \quad \pi \sim \frac{1}{M_P^{D-1}M^2} \hat{\pi}\)
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\]

First non-trivial Non-renormalizable operator appears at \( \Lambda_3 \). Decoupling limit:

\[
M \to 0, \quad M_P \to \infty, \quad \Lambda_{D+2} \text{ fixed} \quad \quad \Lambda_{\frac{D+2}{D-2}} = \left( M^{\frac{1}{D-2}} M_P \right)^{\frac{D-2}{D+2}}
\]

\[
M_P^{D-2} \int d^D x \left[ \frac{1}{8} h^{\mu\nu}(\mathcal{E}h)_{\mu\nu} - \frac{1}{2} f^{\mu\nu}(\mathcal{E}h)_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 - 2M^2 f^{\mu\nu}(\partial_\mu \partial_\nu \pi - \eta_{\mu\nu} \Box \pi) + 2M^2 R^{L \mu}_{\nu \lambda}(h) \partial^\mu \pi \partial^\nu \pi \right]
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\]

\[
M_P^{D-2} \int d^D x \left[ \frac{1}{8} h^{\mu\nu} (\mathcal{E}h)_{\mu\nu} - \frac{1}{2} f^{\mu\nu} (\mathcal{E}h)_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 - 2M^2 f^{\mu\nu} (\partial_\mu \partial_\nu \pi - \eta_{\mu\nu} \Box \pi) + 2M^2 R_{\mu\nu}(h) \partial^\mu \pi \partial^\nu \pi \right]
\]

Diagonalize:

\[
h_{\mu\nu} \to 2 \left(h'_{\mu\nu} + f'_{\mu\nu}\right) - \frac{4}{D-2} M^2 \eta_{\mu\nu} \pi,
\]

\[
f_{\mu\nu} \to f'_{\mu\nu} - \frac{2}{D-2} M^2 \eta_{\mu\nu} \pi - 2M^2 \left[ \partial_\mu \pi \partial_\nu \pi - \frac{1}{D-2} (\partial \pi)^2 \eta_{\mu\nu} \right]
\]

\[
M_P^{D-2} \int d^D x \left[ \frac{1}{2} h'^{\mu\nu} (\mathcal{E}h')_{\mu\nu} - \frac{1}{2} f'^{\mu\nu} (\mathcal{E}f')_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + \frac{2(D-1)}{D-2} M^4 (\partial \pi)^2 - \frac{2M^4(D-4)}{D-2} (\partial \pi)^2 \Box \pi \right]
\]

only interaction is a cubic galileon
Decoupling limit

Canonically normalize:
\[
(h_{\mu\nu}, f_{\mu\nu}) \sim \frac{1}{M_P^{\frac{D-4}{2}}} (\hat{h}_{\mu\nu}, \hat{f}_{\mu\nu}), \quad V_\mu \sim \frac{1}{M_P^{D-1}} \hat{V}_\mu, \quad \pi \sim \frac{1}{M_P^{\frac{D-4}{2}} M^2} \hat{\pi}
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First non-trivial Non-renormalizable operator appears at \(\Lambda_3\). Decoupling limit:

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M \to 0, \quad M_P \to \infty, \quad \Lambda_{\frac{D+2}{D-2}} \text{ fixed} \quad \quad \Lambda_{\frac{D+2}{D-2}} = \left( M \frac{1}{D-2} M_P \right)^{\frac{D-2}{D+2}}
\]

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M_P^{\frac{D-2}{2}} \int d^D x \left[ \frac{1}{8} h^{\mu\nu} (\mathcal{E} h)_{\mu\nu} - \frac{1}{2} f^{\mu\nu} (\mathcal{E} h)_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 - 2M^2 f^{\mu\nu} (\partial_\mu \partial_\nu \pi - \eta_{\mu\nu} \Box \pi) + 2M^2 R^L_{\mu\nu} (h) \partial^\mu \pi \partial^\nu \pi \right]
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Diagonalize:

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h_{\mu\nu} \to 2 (h'_{\mu\nu} + f'_{\mu\nu}) - \frac{4}{D-2} M^2 \eta_{\mu\nu} \pi,
\]

\[
f_{\mu\nu} \to f'_{\mu\nu} - \frac{2}{D-2} M^2 \eta_{\mu\nu} \pi - 2M^2 \left[ \partial_\mu \pi \partial_\nu \pi - \frac{1}{D-2} (\partial \pi)^2 \eta_{\mu\nu} \right]
\]

\[
M_P^{\frac{D-2}{2}} \int d^D x \left[ \frac{1}{2} h'^{\mu\nu} (\mathcal{E} h')_{\mu\nu} - \frac{1}{2} f'^{\mu\nu} (\mathcal{E} f')_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + \frac{2(D-1)}{D-2} M^4 (\partial \pi)^2 \right. 
\]

\[
\left. - \frac{2M^4 (D-4)}{D-2} (\partial \pi)^2 \Box \pi \right]
\]

Becomes strongly coupled at dRGT scale when \(D \neq 4\):

- New Massive gravity non-renormalizable (\(D=3\))
- Quadratic curvature gravity non-renormalizable in \(D \geq 5\)

**only interaction is a cubic galileon**
D=4 massless limit

Galileon term vanishes when $D=4$: we must look for scales higher than $\Lambda_3$.
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It turns out there are none.

Field re-definition removes all operators suppressed by scales $< M_P$

$$f_{\mu\nu} \rightarrow f'_{\mu\nu} + \frac{1}{2} g_{\mu\nu} - 2M^2 \left[ \tilde{V}_\mu \tilde{V}_\nu - \frac{1}{2} g_{\mu\nu} \tilde{V}^2 \right]$$
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No obstruction to taking a straight massless $M \rightarrow 0$ limit:

$$M_P^2 \int d^4 x \sqrt{-g} \left[ - \frac{1}{2} M^2 F_{\mu\nu}^2 + 3M^4 (\partial \pi)^2 
+ f'_{\mu\nu} \left( G_{\mu\nu} - 2M^2 (\nabla_\mu \nabla_\nu \pi - g_{\mu\nu} \Box \pi) + 2M^4 \left( \nabla_\mu \nabla_\nu \pi + \frac{1}{2} g_{\mu\nu} (\partial \pi)^2 \right) \right) \right]$$

Weyl transformation: $g_{\mu\nu} \rightarrow e^{-2M^2 \pi} g_{\mu\nu}$

$$M_P^2 \int d^4 x \sqrt{-g} \left[ f'_{\mu\nu} G_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + 3M^4 e^{-2M^2 \pi} (\partial \pi)^2 \right]$$
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$$f_{\mu\nu} \rightarrow f_{\mu\nu}' + \frac{1}{2}g_{\mu\nu} - 2M^2 \left[ \tilde{V}_\mu \tilde{V}_\nu - \frac{1}{2}g_{\mu\nu}\tilde{V}^2 \right]$$

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+ f'_{\mu\nu} \left( G_{\mu\nu} - 2M^2(\nabla_\mu \nabla_\nu \pi - g_{\mu\nu} \square \pi) + 2M^4 \left( \nabla_\mu \nabla_\nu \pi + \frac{1}{2}g_{\mu\nu}(\partial \pi)^2 \right) \right) \right]$$

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If the theory is renormalizable, there should be no strong coupling at all, even at $M_P$
**D=4 massless limit**

Expand in fluctuations and diagonalize:

\[ h_{\mu\nu} = \tilde{h}_{\mu\nu} + \tilde{f}_{\mu\nu}, \quad \delta f_{\mu\nu} = f_{\mu\nu} - \frac{1}{2} \tilde{h}_{\mu\nu} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad f'_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu} + \delta f_{\mu\nu} \]

\[
M_P^2 \int d^4x \left[ \frac{3}{8} \tilde{h}^{\mu\nu} (\mathcal{E} \tilde{h})_{\mu\nu} - \frac{3}{8} \tilde{f}^{\mu\nu} (\mathcal{E} \tilde{f})_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + 3 M^4 e^{-2M^2 \pi} (\partial \pi)^2 \\
+ \left( f_{\mu\nu} - \frac{1}{2} \tilde{h}_{\mu\nu} \right) \sqrt{-g} G^{(\geq 2)\mu\nu} \left[ \tilde{h} + \tilde{f} \right] + \mathcal{L}^{(\geq 1)}_{V \pi} \left[ \tilde{h} + \tilde{f}, V, \pi \right] \right]
\]
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\[
M_P^2 \int d^4x \left[ \frac{3}{8} \tilde{h}^{\mu\nu} \left( \mathcal{E} \tilde{h} \right)_{\mu\nu} - \frac{3}{8} \tilde{f}^{\mu\nu} \left( \mathcal{E} \tilde{f} \right)_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + 3M^4 e^{-2M^2(\partial \pi)^2} \right.
\]
\[
+ \left( \tilde{f}_{\mu\nu} - \frac{1}{2} \tilde{h}_{\mu\nu} \right) \sqrt{-g} G^{(\geq 2)\mu\nu} \left[ \tilde{h} + \tilde{f} \right] + \mathcal{L}^{(\geq 1)}_{V, \pi} \left[ \tilde{h} + \tilde{f}, V, \pi \right] \]

Coupling is only through the combination \( h + f \)

\[
\begin{align*}
\tilde{h} - \frac{i}{p^2} & \quad + \quad \tilde{f} + \frac{i}{p^2} \\
\text{graviton exchange cancels ghost exchange} & \quad = \quad 0
\end{align*}
\]
Field re-definition completely eliminates interactions:

\[
\begin{pmatrix}
\tilde{h}_{\mu\nu} \\
\tilde{f}_{\mu\nu}
\end{pmatrix} = \begin{pmatrix}
cosh \alpha & \sinh \alpha \\
\sinh \alpha & \cosh \alpha
\end{pmatrix} \begin{pmatrix}
\tilde{h}^{(\alpha)}_{\mu\nu} \\
\tilde{f}^{(\alpha)}_{\mu\nu}
\end{pmatrix}
\]

exploit so(1,1) invariance of ghostly kinetic terms

\[
M_P^2 \int d^4x \left[ \frac{3}{8} \tilde{h}^{(\alpha)\mu\nu} (\mathcal{E}\tilde{h}^{(\alpha)})_{\mu\nu} - \frac{3}{8} \tilde{f}^{(\alpha)\mu\nu} (\mathcal{E}\tilde{f}^{(\alpha)})_{\mu\nu} - \frac{1}{2} M^2 F^2_{\mu\nu} + 3M^4 e^{-2M^2 \pi (\partial\pi)^2} \\
+ \left( \tilde{f} - \frac{1}{2} \tilde{h} \right)_{\mu\nu} \sqrt{-g} G^{(\geq 2)\mu\nu} \left[ e^\alpha \left( \tilde{h}^{(\alpha)} + \tilde{f}^{(\alpha)} \right) \right] + \mathcal{L}_{V,\pi}^{(\geq 1)} \left[ e^\alpha \left( \tilde{h}^{(\alpha)} + \tilde{f}^{(\alpha)} \right), V, \pi \right] \right]
\]
D=4 massless limit

Field re-definition completely eliminates interactions:

\[
\begin{pmatrix}
\tilde{h}_{\mu\nu} \\
\tilde{f}_{\mu\nu}
\end{pmatrix} =
\begin{pmatrix}
\cosh \alpha & \sinh \alpha \\
\sinh \alpha & \cosh \alpha
\end{pmatrix}
\begin{pmatrix}
\tilde{h}^{(\alpha)}_{\mu\nu} \\
\tilde{f}^{(\alpha)}_{\mu\nu}
\end{pmatrix}
\]

exploit so(1,1) invariance of ghostly kinetic terms

\[
M_P^2 \int d^4 x \left[ \frac{3}{8} \tilde{h}^{(\alpha)\mu\nu} (\mathcal{E} \tilde{h}^{(\alpha)})_{\mu\nu} - \frac{3}{8} \tilde{f}^{(\alpha)\mu\nu} (\mathcal{E} \tilde{f}^{(\alpha)})_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + 3M^4 e^{-2M^2 \pi (\partial \pi)^2} \\
+ \left( \tilde{f} - \frac{1}{2} \tilde{h} \right)_{\mu\nu} \sqrt{-g} G^{(2)\mu\nu} \left[ e^\alpha (\tilde{h}^{(\alpha)} + \tilde{f}^{(\alpha)}) \right] + \mathcal{L}_{V,\pi}^{(\geq 1)} \left[ e^\alpha (\tilde{h}^{(\alpha)} + \tilde{f}^{(\alpha)}), V, \pi \right] \right]
\]

\[\alpha \to -\infty \quad \Rightarrow \quad \text{Theory is trivial at high energies (renormalizable & asymptotically free)}\]

\[
M_P^2 \int d^4 x \frac{3}{8} \tilde{h}^{(\alpha)\mu\nu} (\mathcal{E} \tilde{h}^{(\alpha)})_{\mu\nu} - \frac{3}{8} \tilde{f}^{(\alpha)\mu\nu} (\mathcal{E} \tilde{f}^{(\alpha)})_{\mu\nu} - \frac{1}{2} M^2 F_{\mu\nu}^2 + 3M^4 e^{-2M^2 \pi (\partial \pi)^2}
\]
D=4 massless limit

Bringing back the scalar:

\[ M_P^2 \int d^4x \left[ \frac{3}{4} e^{-\pi (\partial \pi)^2} - \frac{3}{4} e^{-\pi (\partial \psi)^2} - \frac{3}{4} m^2 e^{-2(\psi + \pi)} (e^\psi - 1)^2 \right] \]

We can field re-define to find an explicitly renormalizable interaction

\[ \pi = -\log \left( \tilde{\pi}^2 - \tilde{\psi}^2 \right), \quad \psi = \log \left( \frac{\tilde{\pi} + \tilde{\psi}}{\tilde{\pi} - \tilde{\psi}} \right) \]

\[ 3M_P^2 \int d^4x (\partial \tilde{\pi})^2 - (\partial \tilde{\psi})^2 + m^2 \tilde{\psi}^2 \left( \tilde{\pi} - \tilde{\psi} \right)^2 \]

Renormalizable \( \phi^4 \)-type potential with coupling \( \lambda \sim \frac{m^2}{M_P^2} \)
Conclusions

• Quadratic curvature gravity in the high-energy limit is greatly simplified using the Stückelberg trick

• Makes the renormalizability/asymptotic freedom of the theory easy to see

• Shows how the massive graviton overcomes the $\Lambda_3$ strong coupling scale

• Shows the necessity of the ghost in making this possible