Exploring theories beyond Horndeski

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Introduction

- **Viable scalar theories with higher derivative terms ?**
- Main obstacle: **Ostrogradski**'s instabilities

  Theories with higher time derivatives often contain an extra DOF, leading to ghost-like instabilities.

  $$\text{e.g.} \quad L(q, \dot{q}, \ddot{q})$$

- Way out: **degenerate** theories, i.e. such that $\frac{\partial L}{\partial \ddot{q}} = 0$
- Example: galileon models — Nicolis, Rattazzi & Trincherini 08

  They give 2^{nd} order equations of motion

- Scalar-tensor theories ?

Horndeski theories

- Most general scalar-tensor action leading to at most second order equations of motion for the scalar field and metric

Horndeski 74

- Rediscovered as « Generalized Galileons », with the requirement that equations of motion are 2nd order

Deffayet, Gao, Steer & Zahariade 11

- Combination of the following four Lagrangians

$$X \equiv \nabla_\mu \phi \nabla^\mu \phi$$

$$\phi_{\mu\nu} \equiv \nabla_\nu \nabla_\mu \phi$$

$$L_2^H = G_2(\phi, X)$$

$$L_3^H = G_3(\phi, X)\,\Box\phi$$

$$L_4^H = G_4(\phi, X)\,{}^{(4)}R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})$$

$$L_5^H = G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})$$

Beyond Horndeski

Gleyzes, DL, Piazza & Vernizzi '14

- 2nd order equations of motion are in fact not necessary
- Extended class of Lagrangians (dubbed G3)

$$L_2^H = G_2(\phi, X)\ , \qquad L_3^H = G_3(\phi, X)\,\Box\phi\ ,$$

$$L_4^H = G_4(\phi, X)\,{}^{(4)}R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})$$

$$L_4^{bH} = F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\,\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\ ,$$

$$L_5^H = G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})$$

$$L_5^{bH} = F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'}$$

with two new terms that do not belong to Horndeski class.

The Fab Four

Charmousis, Copeland, Padilla & Saffin '11

- **Subclass of Horndeski** theories that satisfy **self-tuning**
  - there exists a Minkowski solution for any cosmological constant
  - this remains true if the cosmological constant changes suddenly
  - non trivial cosmological solutions exist

- Fab Four Lagrangians

$$\mathcal{L}_{john} = \sqrt{-g}\, V_J(\phi)\, G^{\mu\nu}\, \nabla_\mu\phi\nabla_\nu\phi\,,$$

$$\mathcal{L}_{paul} = \sqrt{-g}\, V_P(\phi)\, P^{\mu\nu\alpha\beta}\, \nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi\,,$$

$$\mathcal{L}_{george} = \sqrt{-g}\, V_G(\phi)\, R\,,$$

$$\mathcal{L}_{ringo} = \sqrt{-g}\, V_R(\phi)\, \hat{G}\,.$$

$$P_{\alpha\beta\mu\nu} \equiv -\frac{1}{4}\epsilon_{\alpha\beta\rho\sigma}R^{\rho\sigma\gamma\delta}\epsilon_{\mu\nu\gamma\delta}$$

Beyond the Fab Four

Babichev, Charmousis, DL & Saito '15

- The new Lagrangians beyond Horndeski, L_4^{bH} and L_5^{bH}, can be rewritten as **extended John and Paul terms**

$$S_J^{\mathrm{ext}} = \int d^4x\, \sqrt{-g}\, F_J(\phi, X)\, G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

$$= \int d^4x\, \sqrt{-g}\, \left\{F_{J,X}\, \epsilon_{\mu\gamma\alpha\beta}\, \epsilon_{\nu\delta\rho}{}^{\beta}\, \phi^{;\mu}\phi^{;\nu}\phi^{;\delta\gamma}\phi^{;\alpha\rho} + \text{Horndeski terms}\right\}$$

$$S_P^{\mathrm{ext}} = \int\, d^4x\, \sqrt{-g}\, F_P(\phi, X)\, P^{\alpha\beta\mu\nu}\nabla_\alpha\phi\nabla_\mu\phi\nabla_\beta\nabla_\nu\phi$$

$$= \int d^4x\, \sqrt{-g}\, \left\{-\tfrac{F_{P,X}}{3}\, \epsilon_{\mu\gamma\alpha\beta}\, \epsilon_{\nu\delta\rho\sigma}\, \phi^{;\mu}\phi^{;\nu}\phi^{;\delta\gamma}\phi^{;\alpha\rho}\phi^{;\beta\sigma} + \text{Horndeski terms}\right\}$$

- **Same self-tuning properties**

Degenerate higher derivative scalar-tensor theories

DL & K. Noui 2015

Toy model

- Consider the model

$$L = \frac{1}{2} a \, \ddot{\phi}^2 + \frac{1}{2} k_0 \dot{\phi}^2 - V(\phi)$$

If $a \neq 0$, the Lagrangian is **nondegenerate.**

  – The equation of motion is fourth order
  – The system contains 2 degrees of freedom (phase space is 4 dimensional)
  – Ostrogradski instability

- To get rid of the Ostrogradski ghost requires $a = 0$

Toy model

- What if the system is coupled to other dof ?

$$L = \frac{1}{2} a\,\ddot{\phi}^2 + \frac{1}{2} k_0 \dot{\phi}^2 + \frac{1}{2} k_{ij} \dot{q}^i \dot{q}^j + b_i\,\ddot{\phi}\,\dot{q}^i - V(\phi, q)$$

$$i = 1, \ldots, n$$

- Equations of motion

$$a\,\ddddot{\phi} - k_0 \ddot{\phi} + b_i\,\dddot{q}^i - c_i \ddot{q}^i - V_\phi = 0\ ,$$

$$k_{ij} \ddot{q}^j + b_i\,\dddot{\phi} + c_i \ddot{\phi} + V_i = 0$$

  - 4th order if $a \neq 0$
  - 3rd order if $a = 0\,, \quad b_i \neq 0$
  - 2nd order if $a = 0\,, \quad b_i = 0$

Degrees of freedom

- **Number of degrees of freedom ?**

  Let us rewrite the Lagrangian in the form

$$L = \frac{1}{2} a\,\dot{Q}^2 + \frac{1}{2} k_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2} k_0 Q^2 + b_i \dot{Q}\dot{q}^i - V(\phi, q) - \lambda(Q - \dot{\phi})$$

- **Equations of motion**

$$a\ddot{Q} + b_i \ddot{q}^i = k_0 Q - \lambda$$
$$b_i \ddot{Q} + k_{ij}\ddot{q}^j = -V_i$$

$$\dot{\phi} = Q\,, \quad \dot{\lambda} = -V_\phi$$

If the **kinetic matrix** $M = \begin{pmatrix} a & b_j \\ b_i & k_{ij} \end{pmatrix}$ is **invertible**, one needs 2(n+1)+2 initial conditions.

n+2 DOF, including the Ostrogradski ghost !

Degrees of freedom

- **Degenerate case**

Degenerate kinetic matrix $M = \begin{pmatrix} a & b_j \\ b_i & k_{ij} \end{pmatrix}$

$$\det(M) = \det(k)\left(a - b_i b_j (k^{-1})^{ij}\right) = 0$$

There are (at most) n+1 degrees of freedom

- Eqs of motion can be rewritten as a 2nd order system for ϕ and the new variables $x^i \equiv q^i + (k^{-1})^{ij} b_j\, Q$
- Hamiltonian analysis (primary constraint, which generates a secondary constraint)

No Ostrogradski ghost ! (with eom's of order 2, 3 or 4)

Scalar-tensor theories

- We consider all Lagrangians of the form

$$S[g,\phi] \equiv \int \sqrt{-g}\,(f\,\mathcal{R} + C^{\mu\nu\rho\sigma}\,\nabla_\mu\nabla_\nu\phi\,\nabla_\rho\nabla_\sigma\phi)$$

where $f = f(X,\phi)$ and $C^{\mu\nu\rho\sigma}$ depends only on ϕ and $\nabla_\mu\phi$.

- Symmetries: $C^{\mu\nu\rho\sigma} = C^{\nu\mu\rho\sigma} = C^{\mu\nu\sigma\rho} = C^{\rho\sigma\mu\nu}$

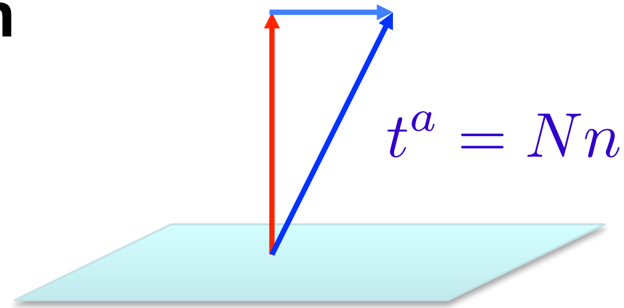
$$C^{\mu\nu\rho\sigma} = \alpha_1\,g^{\mu\nu}g^{\rho\sigma} + \alpha_2\,g^{\mu(\rho}g^{\sigma)\nu} + \frac{1}{2}\alpha_3\,(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu})$$
$$+\frac{1}{4}\alpha_4(\phi^\mu\phi^\rho g^{\nu\sigma} + \ldots) + \alpha_5\,\phi^\mu\phi^\nu\phi^\rho\phi^\sigma$$

- This includes the Lagrangians L_4^{H} and L_4^{bH}

Scalar-tensor theories

- Introduce the auxiliary field $A_b = \nabla_b \phi$

- **Covariant 3+1 decomposition**



$$t^a = N n^a + N^a$$

$$h_{ab} \equiv g_{ab} + n_a n_b$$

- Normal and spatial components of A_a

$$A_* \equiv A_a n^a \qquad \left[A_* = (\dot{\phi} - N^k A_k)/N\right]$$

$$\hat{A}_a \equiv h_a^b A_b$$

Scalar-tensor theories

- 3+1 decomposition of $\nabla_a A_b$

$$(\nabla_a A_b)_{\rm kin} = \lambda_{ab}\,\dot{A}_* + \Lambda_{ab}{}^{cd}\,K_{cd}$$

with $K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - D_a N_b - D_b N_a\right)$.

$$\lambda_{ab} \equiv \frac{1}{N} n_a n_b \qquad \Lambda_{ab}{}^{cd} \equiv -A_*\, h^c_{(a} h^d_{b)} + 2\, n_{(a} h^{(c}_{b)} \hat{A}^{d)}$$

- Substituting into the total Lagrangian, one gets

$$L_{\rm kin} = \mathcal{A}\,\dot{A}_*{}^2 + 2\,\mathcal{B}^{ef}\,\dot{A}_* K_{ef} + \mathcal{K}^{ab,cd}\,K_{ef}\,K_{gh}$$

Scalar-tensor theories

- The kinetic part of the Lagrangian

$$L_{\rm kin} = \mathcal{A}\, \dot{A}_*^{\;2} + 2\, \mathcal{B}^{ef}\, \dot{A}_* K_{ef} + \mathcal{K}^{ab,cd}\, K_{ef}\, K_{gh}$$

yields the kinetic matrix $\mathcal{M} = \begin{pmatrix} \mathcal{A} & \mathcal{B}^{cd} \\ \mathcal{B}^{ab} & \mathcal{K}^{ab,cd} \end{pmatrix}$

- **Degeneracy**: eigenvector $\begin{pmatrix} v_0 \\ \mathcal{V}_{cd} \end{pmatrix}$ with eigenvalue 0

$$v_0\, \mathcal{A} + \mathcal{B}^{cd} \mathcal{V}_{cd} = 0\,, \qquad v_0\, \mathcal{B}^{ab} + \mathcal{K}^{ab,cd}\, \mathcal{V}_{cd} = 0$$

$$\mathcal{V}_{cd} = v_1\, h_{cd} + v_2\, \hat{A}_c \hat{A}_d$$

Scalar-tensor theories

- Degenerate if the determinant vanishes:

$$D_0(X) + D_1(X)A_*^2 + D_2(X)A_*^4 = 0$$

$$D_0(X) \equiv -4(\alpha_1+\alpha_2)\left[Xf(2\alpha_2 + X\alpha_4 + 4f_X) - 2f^2 - 8X^2 f_X^2\right]$$

$$\begin{aligned} D_1(X) \equiv\ & 4\left[X^2\alpha_2(3\alpha_1+\alpha_2) - 2f^2 - 4Xf\alpha_1\right]\alpha_4 + 4X^2 f(\alpha_1+\alpha_2)\alpha_5 \\ & +8X\alpha_2^3 - 4(f + 4Xf_X - 6X\alpha_1)\alpha_2^2 - 16(f+5Xf_X)\alpha_1\alpha_2 \\ & +4X(3f - 4Xf_X)\alpha_2\alpha_3 - X^2 f\alpha_3^2 + 32f_X(f+2Xf_X)\alpha_1 \\ & -16ff_X\alpha_2 - 8f(f - Xf_X)\alpha_3 + 48ff_X^2 \end{aligned}$$

$$\begin{aligned} D_2(X) \equiv\ & 4\left[2f^2 + 4Xf\alpha_1 - X^2\alpha_2(3\alpha_1+\alpha_2)\right]\alpha_5 + 4\alpha_2^3 \\ & +4(2\alpha_1 - X\alpha_3 - 4f_X)\alpha_2^2 + 3X^2\alpha_2\alpha_3^2 - 4Xf\alpha_3^2 \\ & +8(f+Xf_X)\alpha_2\alpha_3 - 32f_X\alpha_1\alpha_2 + 16f_X^2\alpha_2 + 32f_X^2\alpha_1 - 16ff_X\alpha_3 \end{aligned}$$

Classification of the degenerate Lagrangians

- **Degeneracy conditions**

$$D_0(X) = 0, \quad D_1(X) = 0, \quad D_2(X) = 0$$

- Case $\mathcal{A} = 0$ (degenerate for nondynamical metric)

$$\mathcal{A} = \frac{1}{N^2}\left[\alpha_1 + \alpha_2 - (\alpha_3 + \alpha_4)A_*^2 + \alpha_5 A_*^4\right]$$

$$\alpha_1 + \alpha_2 = 0\,, \quad \alpha_3 + \alpha_4 = 0\,, \quad \alpha_5 = 0$$

All degenerate theories of this type coincide with quartic G3 (including Horndeski)

$$f = G_4\,, \quad \alpha_1 = -\alpha_2 = -2G_{4X} - XF_4\,, \quad \alpha_3 = -\alpha_4 = 2F_4$$

Classification of the degenerate Lagrangians

- Case $\mathcal{A} \neq 0$

Three families of Lagrangians that depend on 3 arbitrary functions (f, α_1, α_3)

- subcase $\alpha_1 + \alpha_2 = 0$

$$\alpha_4 = \frac{16X\alpha_1^3 + 12\alpha_1^2 - 12X\alpha_3\alpha_1 - X^2\alpha_3^2 - 8\alpha_3}{8(1+X\alpha_1)^2}, \quad \alpha_5 = \frac{(2\alpha_1 + X\alpha_3)\left(4\alpha_3 - 2\alpha_1^2 + 3X\alpha_1\alpha_3\right)}{8(1+X\alpha_1)^2}$$

- subcases $\alpha_1 + \alpha_2 \neq 0$

$$\alpha_2 = \frac{1}{X}, \quad \alpha_4 = 0, \quad \alpha_5 = \frac{-4 - 8X\alpha_1 - 4X^2\alpha_3 + X^4\alpha_3^2}{4X^3(1+X\alpha_1)}$$

$$\alpha_2 = -\frac{2}{X} - \frac{X}{2}\alpha_3 - 4\alpha_1, \quad \alpha_4 = \frac{6}{X^2} + \alpha_3 + \frac{8}{X}\alpha_1, \quad \alpha_5 = -\frac{4 + 8X\alpha_1 + 3X^2\alpha_3}{X^3}$$

Conclusions

- Systematic classification of “degenerate” theories
- The quartic G^3 terms describe the most general degenerate theories both with and without gravity
- Three new families of degenerate Lagrangians, which are nondegenerate with a nondynamical metric
- We conjecture that all these degenerate theories do not suffer from Ostrogradski’s instability
- For future work:
  - Hamiltonian formulation
  - Lagrangians cubic in $\phi_{\mu\nu}$