Perturbations of Cosmological and Black Hole Solutions in Massive gravity and Bi-gravity

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$c = \hbar = 1$

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Introduction

(Theoretical) motivation of Massive gravity

- General relativity (GR)

  The theory of an interacting massless helicity 2 particle

- Massive gravity

  A theory of an interacting massive spin 2 particle

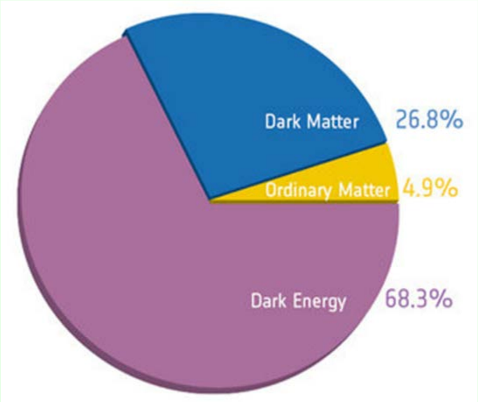
What is it ?

(Observational) motivation of Massive gravity

The Universe is now accelerating !!

➡ {
- **Dark Energy is introduced**

  or
- **GR may be modified in the IR limit**



PLANCK

One possibility ⬇ **Massive gravity**

If the graviton has a mass comparable to the present Hubble scale, gravity is suppressed beyond that scale.

➔ the present Universe looks accelerating.

Massive gravity mimics GR with C.C.

At the background level, we are interested in the following solutions of massive gravity (bigravity) :

$$G^{\mu}{}_{\nu} + \Lambda \delta^{\mu}{}_{\nu} = \frac{1}{M_{\rm pl}^2} T^{\mu}{}_{\nu}$$

coming from the mass term

Can we always discriminate massive gravity (bigravity) from GR at the perturbation level ?

I am going to address this issue in this talk.

Basics of Massive Gravity (Bigravity)

(Please see Hinterbichler 2012 and de Rham 2014 for good review)

Action of massive gravity (bigravity)

de Rham & Gabadadze 2010
de Rham, Gabadadze, Tolley 2011
Hassan & Rosen 2012

$g_{\mu\nu}$: physical metric, $f_{\mu\nu}$: fiducial metric ➔ $\gamma^{\mu}{}_{\nu} = \left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\nu}$

$$S = \frac{1}{2}M_{\rm pl}^2 \int d^4x\sqrt{-g}R[g] + \int d^4x\sqrt{-g}\,\mathcal{L}^{(g)}_{\rm matter}$$
$$+\frac{1}{2}\kappa^2 M_{\rm pl}^2 \int d^4x\sqrt{-f}R[f] + \int d^4x\sqrt{-f}\,\mathcal{L}^{(f)}_{\rm matter}$$
$$+S_{\rm mass}[g,f].$$

(Matter couplings with two general covariance)

$$S_{\rm mass}[g,f] = \frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{-g}\,2\sum_{i=0}^{4}\beta_i e_i(\gamma)$$

$$\begin{cases} e_0(\gamma) = 1, \\ e_1(\gamma) = {\rm Tr}[\gamma], \\ e_2(\gamma) = \frac{1}{2}\left({\rm Tr}[\gamma]^2 - {\rm Tr}[\gamma^2]\right), \\ e_3(\gamma) = \frac{1}{3!}\left({\rm Tr}[\gamma]^3 - 3{\rm Tr}[\gamma]{\rm Tr}[\gamma^2] + 2{\rm Tr}[\gamma^3]\right), \\ e_4(\gamma) = \det(\gamma). \end{cases}$$

$$= \frac{M_{\rm pl}^2}{2}\int d^4x\left[\sqrt{-g}2m^2\sum_{i=2}^{4}\alpha_i e_i(\mathcal{K}) + \sqrt{-g}(-2\Lambda^{(g)}) + \sqrt{-f}(-2\kappa^2\Lambda^{(f)})\right].$$

$$\left(\mathcal{K}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \gamma^{\mu}{}_{\nu}\right)$$

$$\begin{cases} \beta_0 = -\Lambda^{(g)} + m^2(6 + 4\alpha_3 + \alpha_4), \\ \beta_1 = m^2(-3 - 3\alpha_3 - \alpha_4), \\ \beta_2 = m^2(1 + 2\alpha_3 + \alpha_4), \\ \beta_3 = m^2(-\alpha_3 - \alpha_4), \\ \beta_4 = -\kappa^2\Lambda^{(f)} + m^2\alpha_4. \end{cases}$$

($\alpha_2 = 1$)

5 parameters including two C.C.

(m^2, α_3, α_4, $\Lambda^{(g)}$, $\Lambda^{(f)}$)

Equations of motion

- **Variation of action with respect to $g_{\mu\nu}$:**

$$G[g]^{\mu}{}_{\nu} + X_{(g)}{}^{\mu}{}_{\nu} = \frac{1}{M_{\rm pl}^2} T_{(g)}{}^{\mu}{}_{\nu} \qquad \left(T_{(g)}{}^{\mu}{}_{\nu} = \frac{2}{\sqrt{-g}} g^{\mu\rho} \frac{\delta S_{\rm matter}[g]}{\delta g^{\rho\nu}} \right)$$

$$X_{(g)}{}^{\mu}{}_{\nu} = 2 \left(\tau^{\mu}{}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \sum_{i=0}^{3} \beta_i e_i(\gamma) \right),$$

$$\tau^{\mu}{}_{\nu} = \frac{1}{2} \left[\beta_1 \gamma^{\mu}{}_{\nu} + \beta_2 \left(e_1(\gamma) \gamma^{\mu}{}_{\nu} - (\gamma^2)^{\mu}{}_{\nu} \right) + \beta_3 \left(e_2(\gamma) \gamma^{\mu}{}_{\nu} - e_1(\gamma)(\gamma^2)^{\mu}{}_{\nu} + (\gamma^3)^{\mu}{}_{\nu} \right) \right].$$

- **Variation of action with respect to $f_{\mu\nu}$:**

$$G[f]^{\mu}{}_{\nu} + X_{(f)}{}^{\mu}{}_{\nu} = \frac{1}{\kappa^2 M_{\rm pl}^2} T_{(f)}{}^{\mu}{}_{\nu} \qquad \left(T_{(f)}{}^{\mu}{}_{\nu} = -\frac{2}{\sqrt{-f}} \frac{\delta S_{\rm matter}[f]}{\delta f_{\mu\rho}} f_{\rho\nu} \right)$$

$$X_{(f)}{}^{\mu}{}_{\nu} = -\frac{m^2}{\kappa^2} {\rm sgn}(\det \gamma) \left(\frac{2}{\det \gamma} \tau^{\mu}{}_{\nu} + \beta_4 \delta^{\mu}{}_{\nu} \right).$$

Thanks to the two general covariance of matter actions we assumed,
both energy-momentum tensors are conserved :

$$\nabla^{(g)}_{\mu} T_{(g)}{}^{\mu}{}_{\nu} = 0, \qquad \nabla^{(f)}_{\mu} T_{(f)}{}^{\mu}{}_{\nu} = 0.$$

Background solutions

Background metric

We are mainly interested in cosmological and black hole solutions.

(Too many references, so please see the references in our paper)

Bi-spherically symmetric background metric:

$$\begin{cases} \bar{g}_{\mu\nu}dx^\mu dx^\nu = \bar{g}_{tt}(t,r)dt^2 + 2\bar{g}_{tr}(t,r)dtdr + \bar{g}_{rr}(t,r)dr^2 + R(t,r)^2 d\Omega^2, \\ \bar{f}_{\mu\nu}dx^\mu dx^\nu = \bar{f}_{tt}(t,r)dt^2 + 2\bar{f}_{tr}(t,r)dtdr + \bar{f}_{rr}(t,r)dr^2 + A^2(t,r)R^2(t,r)d\Omega^2. \end{cases}$$

$\left(d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\right)$

$$(\bar{g}^{-1}\bar{f})^\mu{}_\nu = \begin{pmatrix} (\bar{g}^{-1}\bar{f})^t{}_t & (\bar{g}^{-1}\bar{f})^t{}_r & 0 & 0 \\ (\bar{g}^{-1}\bar{f})^r{}_t & (\bar{g}^{-1}\bar{f})^r{}_r & 0 & 0 \\ 0 & 0 & A^2(t,r) & 0 \\ 0 & 0 & 0 & A^2(t,r) \end{pmatrix}.$$

$$\bar{\gamma}^\mu{}_\nu = \left(\sqrt{\bar{g}^{-1}\bar{f}}\right)^\mu{}_\nu = \begin{pmatrix} a(t,r) & b(t,r) & 0 & 0 \\ c(t,r) & d(t,r) & 0 & 0 \\ 0 & 0 & A(t,r) & 0 \\ 0 & 0 & 0 & A(t,r) \end{pmatrix}.$$

In the following discussion, the concrete expressions of a, b, c, d are unnecessary.

Mimic cosmological constant

At the background level, we are interested in the case where the correction term in EOM of $g_{\mu\nu}$ reduces to cosmological constant.

$$G[g]^{\mu}{}_{\nu} + \Lambda\delta^{\mu}{}_{\nu} = \frac{1}{M_{\rm pl}^2} T_{(g)}{}^{\mu}{}_{\nu}$$

$$\bar{X}_{(g)}{}^{t}{}_{r} = -m^2 b[3 - 2A + (A-3)(A-1)\alpha_3 + (A-1)^2\alpha_4] = 0,$$
$$\bar{X}_{(g)}{}^{r}{}_{t} = -m^2 c[3 - 2A + (A-3)(A-1)\alpha_3 + (A-1)^2\alpha_4] = 0.$$

We focus on the case with $A(t,r) = (2\alpha_3 + \alpha_4 + 1 \pm \sqrt{\alpha_3^2 + \alpha_3 - \alpha_4 + 1})/(\alpha_3 + \alpha_4) = \text{const.}$

$$\bar{X}_{(g)}{}^{t}{}_{t} - \bar{X}_{(g)}{}^{\theta}{}_{\theta} = (1-A)C(t,r)\,[A - 2 + (A-1)\alpha_3] = 0.$$

$$\left(C(t,r) = m^2 \frac{A^2 - A(a+d) + ad - bc}{(1-A)^2}\right)$$

We focus on the case with $A(t,r) = (2+\alpha_3)/(1+\alpha_3) = \text{const.}$

These two requirements lead to the relation between α3 & α4 :

$$0 = 1 + \alpha_3 + \alpha_3^2 - \alpha_4 \;\left(= \beta_2^2 - \beta_1\beta_3\right)$$

(Chamseddine & Volkov 2011, Gratia, Hu, Wyman 2012, Kobayashi, Siino, MY, Yoshida 2012, Nieuwenhuizen 2011, Berezhani et al. 2012 …)

Background solution

$$\begin{cases} \bar{g}_{\mu\nu}dx^\mu dx^\nu = \bar{g}_{tt}(t,r)dt^2 + 2\bar{g}_{tr}(t,r)dtdr + \bar{g}_{rr}(t,r)dr^2 + R(t,r)^2 d\Omega^2, \\ \bar{f}_{\mu\nu}dx^\mu dx^\nu = \bar{f}_{tt}(t,r)dt^2 + 2\bar{f}_{tr}(t,r)dtdr + \bar{f}_{rr}(t,r)dr^2 + A^2(t,r)R^2(t,r)d\Omega^2. \end{cases}$$

with $\begin{cases} A(t,r) = (2+\alpha_3)/(1+\alpha_3) = \text{const.} \\ 0 = 1 + \alpha_3 + \alpha_3^2 - \alpha_4 \ \left(= \beta_2^2 - \beta_1\beta_3\right) \end{cases}$

$$\Rightarrow \begin{cases} G[g]^\mu{}_\nu + X_{(g)}{}^\mu{}_\nu = \dfrac{1}{M_{\text{pl}}^2} T_{(g)}{}^\mu{}_\nu. \\ \Downarrow \\ \Lambda_{\text{eff}}^{(g)} \delta^\mu{}_\nu \quad \left(\Lambda_{\text{eff}}^{(g)} = m^2(A-1) + \Lambda^{(g)}\right) \\ G[f]^\mu{}_\nu + X_{(f)}{}^\mu{}_\nu = \dfrac{1}{\kappa^2 M_{\text{pl}}^2} T_{(f)}{}^\mu{}_\nu \\ \Downarrow \\ \Lambda_{\text{eff}}^{(f)} \delta^\mu{}_\nu \quad \left(\Lambda_{\text{eff}}^{(f)} = \text{sgn}(ad-bc)\left(-\dfrac{m^2}{\kappa^2}\dfrac{A-1}{A} + \Lambda^{(f)}\right)\right) \end{cases}$$

Since the equations of motion for $g_{\mu\nu}$ reduce to the Einstein equations with a cosmological constant, any spherically symmetric solution in GR is also a solution of massive gravity (bigravity) with a suitable fiducial metric.

Perturbations

At background level, massive gravity (bigravity) mimics GR with C.C.
Is there always any difference between massive gravity and GR at perturbation level ?
(e.g. cosmological perturbations, quasi-normal modes around BH)

First order perturbations

Linear perturbations around bi-spherically symmetric solutions :

$$\begin{cases} g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \\ f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu}. \end{cases}$$

$$\Rightarrow \quad \sqrt{g^{-1}f} = \gamma^{\mu}{}_{\nu} = \bar{\gamma}^{\mu}{}_{\nu} + \delta\gamma^{\mu}{}_{\nu} + \mathcal{O}(\text{second-order}).$$

$$(\bar{\gamma}^{\mu}{}_{\rho}\delta\gamma^{\rho}{}_{\nu} + \delta\gamma^{\mu}{}_{\rho}\bar{\gamma}^{\rho}{}_{\nu} = -\bar{g}^{\mu\rho}\delta g_{\rho\nu}\bar{\gamma}^{\rho}{}_{\sigma}\bar{\gamma}^{\sigma}{}_{\nu} + \bar{g}^{\mu\rho}\delta f_{\rho\nu})$$

We do not need the explicit form of the solution for δγ.

$$\delta G[g]^{\mu}{}_{\nu} + \delta X_{(g)}{}^{\mu}{}_{\nu} = \frac{1}{M_{\text{pl}}^2}\delta T_{(g)}{}^{\mu}{}_{\nu}.$$

$$\Rightarrow \quad \delta X_{(g)}{}^{\mu}{}_{\nu} = C(t,r)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\gamma^{\phi}{}_{\phi} & -\delta\gamma^{\theta}{}_{\phi} \\ 0 & 0 & -\delta\gamma^{\phi}{}_{\theta} & \delta\gamma^{\theta}{}_{\theta} \end{pmatrix}.$$

Bianchi Identities

Since the Einstein tensor satisfies the Bianchi identity and the energy-momentum tensor is conserved :

$$\nabla^{(g)}_{\mu} X_{(g)}{}^{\mu}{}_{\nu} = -\nabla^{(g)}_{\mu} G[g]^{\mu}{}_{\nu} + \frac{1}{M_{\rm pl}^2} \nabla^{(g)}_{\mu} T_{(g)}{}^{\mu}{}_{\nu} = 0. \quad \Longrightarrow \quad \bar{\nabla}^{(\bar{g})}_{\mu} \delta X_{(g)}{}^{\mu}{}_{\nu} = 0.$$

Inserting $\delta X_{(g)}{}^{\mu}{}_{\nu} = C(t,r) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\gamma^{\phi}{}_{\phi} & -\delta\gamma^{\theta}{}_{\phi} \\ 0 & 0 & -\delta\gamma^{\phi}{}_{\theta} & \delta\gamma^{\theta}{}_{\theta} \end{pmatrix}.$ yields

$$\Longrightarrow \begin{cases} \delta\gamma^{\theta}{}_{\theta} + \delta\gamma^{\phi}{}_{\phi} = 0 & (\nu = t, r \text{ component}) \\ \partial_{\theta}\left(\sin\theta\, \delta\gamma^{\theta}{}_{\phi}\right) = -\partial_{\phi}\left(\sin\theta\, \delta\gamma^{\theta}{}_{\theta}\right) & (\nu = \theta \text{ component}) \\ \boxed{\frac{1}{\sin\theta}\partial_{\theta}\left(\sin\theta\, \partial_{\theta}(\sin^2\theta\, \delta\gamma^{\theta}{}_{\theta})\right) + \frac{1}{\sin\theta^2}\partial_{\phi}\partial_{\phi}\left(\sin^2\theta\, \delta\gamma^{\theta}{}_{\theta}\right) = 0} & (\nu = \varphi \text{ component}) \end{cases}$$

- Laplace equation on sphere $\Longrightarrow$ $\sin^2\theta\, \delta\gamma^{\theta}{}_{\theta} = \text{const}$ on sphere
- Requiring $\delta\gamma^{\theta}{}_{\theta}$ to be regular at θ= 0, π $\Longrightarrow$ $\delta\gamma^{\theta}{}_{\theta} = 0$

$$\Longrightarrow \quad \delta X_{(g)}{}^{\mu}{}_{\nu} = 0. \quad \Longleftrightarrow \quad \delta\gamma^{p}{}_{q} = 0 \quad (p, q = \theta, \phi)$$

From the relation $\delta X_{(f)}{}^{\mu}{}_{\nu} = -\frac{1}{\kappa^2 A^2 |ad - bc|} \delta X_{(g)}{}^{\mu}{}_{\nu},$ $\quad \delta X_{(f)}{}^{\mu}{}_{\nu} = 0.$

Summary of first order perturbations

$$\begin{cases} \delta G[g]^{\mu}{}_{\nu} = \delta T_{(g)}{}^{\mu}{}_{\nu}, \\ \delta G[f]^{\mu}{}_{\nu} = \delta T_{(f)}{}^{\mu}{}_{\nu}, \\ A^2 \delta g_{pq} - \delta f_{pq} = 0. \end{cases} \iff \delta\gamma^{p}{}_{q} = 0 \quad (p, q = \theta, \phi)$$

The equations of motion for the perturbations of the two metrics coincide with the perturbed Einstein equations, though δg_{pq} and δf_{pq} are related.

Gauge symmetry representing choice of coordinate

Graviton degrees of freedom : 10x2 - 4x2 – 3 - (4+1) = 4

(Hamiltonian & momentum constraints) **Additional gauge symmetry**

Coincides with those of two massless gravitons.

(confirmed by Hamilton analysis : 20x2 - 10x2 – 12 = 8 phase space d.o.f)

Additional gauge symmetry

Kodama & Arraut 2014

A combination of gauge transformation of $g_{\mu\nu}$ and $f_{\mu\nu}$ separately but keeping the following condition : $A^2\delta g_{pq} - \delta f_{pq} = 0 \quad (p, q = \theta, \phi).$

Infinitesimal gauge transformation generated by

$$\begin{cases} x^\mu \to x^\mu - \xi^\mu & \text{for } g_{\mu\nu} \\ x^\mu \to x^\mu - (\xi^\mu + \delta\xi^\mu) & \text{for } f_{\mu\nu} \end{cases}$$

$$\Rightarrow \quad A^2\delta g_{pq} - \delta f_{pq} \to A^2\delta g_{pq} - \delta f_{pq} + \Delta_{pq}.$$

Additional gauge symmetry ⇔ $\Delta pq = 0$.

$$\Rightarrow \begin{cases} \delta\xi^0 = \Xi(t, r, \theta, \phi), \\ \delta\xi^1 = -\dfrac{\partial_t R(t,r)\Xi(t,r,\theta,\phi) + R(t,r)Q(t,r)\cos\theta}{\partial_r R(t,r)}, \\ \delta\xi^2 = Q(t,r)\sin\theta, \\ \delta\xi^3 = P(t,r). \end{cases}$$

The quadratic action of the mass term for the linear perturbations :

$$S^{(2)}_{\text{mass}} = \frac{M_{\text{pl}}^2}{2}\int d^4x\sqrt{-\bar{g}}\frac{2C(t,r)}{AR^2}(\delta\gamma^\theta{}_\theta\delta\gamma^\phi{}_\phi - \delta\gamma^\theta{}_\phi\delta\gamma^\phi{}_\theta) + \frac{M_{\text{pl}}^2}{2}\int d^4x\sqrt{-g}^{(2)}(-2\Lambda^{(g)}_{\text{eff}}) + \frac{\kappa^2 M_{\text{pl}}^2}{2}\int d^4x\sqrt{-f}^{(2)}(-2\Lambda^{(f)}_{\text{eff}}).$$

It is manifest that the above action (with EH terms) possesses this symmetry.

Summary of first order perturbations II

The equations of motion for the perturbations of the two metrics coincide with the perturbed Einstein equations at linear order, though δg_{pq} and δf_{pq} are related.

pros and cons :

⇒

- **If background solutions are stable in GR for linear perturbations, so are those in massive gravity (bigravity).**
- **We cannot discriminate massive gravity from GR by use of these solutions.**

⇒ **How about going into second order perturbations ?**

Second order perturbations

Second order perturbations

Second order perturbations around bi-spherically symmetric solutions :

$$\begin{cases} g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} + \delta g^{(2)}{}_{\mu\nu}, \\ f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu} + \delta f^{(2)}{}_{\mu\nu}. \end{cases}$$

$$\Rightarrow \sqrt{g^{-1}f} = \gamma^{\mu}{}_{\nu} = \bar{\gamma}^{\mu}{}_{\nu} + \delta\gamma^{\mu}{}_{\nu} + \delta\gamma^{(2)\mu}{}_{\nu} + \mathcal{O}(\text{third-order}).$$

$$(\delta\gamma^{p}{}_{q} = 0)$$

We do not need the explicit form of the solution for $\delta\gamma^{(2)}$.

$$\delta G[g]^{(2)\,\mu}{}_{\nu} + \delta X_{(g)}{}^{(2)\,\mu}{}_{\nu} = \frac{1}{M_{\rm pl}^2}\delta T_{(g)}{}^{(2)\,\mu}{}_{\nu}.$$

$$\Rightarrow \quad \delta X_{(g)}{}^{(2)\,\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta X_{(g)}{}^{(2)\,\theta}{}_{\theta} & \delta X_{(g)}{}^{(2)\,\theta}{}_{\phi} \\ 0 & 0 & \dfrac{\delta X_{(g)}{}^{(2)\,\theta}{}_{\phi}}{\sin^2\theta} & \delta X_{(g)}{}^{(2)\,\phi}{}_{\phi} \end{pmatrix}.$$

Bianchi Identities

Since the Einstein tensor satisfies the Bianchi identity and the energy-momentum tensor is conserved :

$$\nabla_\mu^{(g)} X_{(g)}{}^\mu{}_\nu = -\nabla_\mu^{(g)} G[g]^\mu{}_\nu + \frac{1}{M_{\rm pl}^2}\nabla_\mu^{(g)} T_{(g)}{}^\mu{}_\nu = 0.$$

$\Longrightarrow$ $\bar{\nabla}_\mu^{(\bar{g})} \delta X_{(g)}{}^{(2)\,\mu}{}_\nu = 0.$

$\left(\because\ \delta X_{(g)}{}^\mu{}_\nu = 0\right)$

Inserting $\delta X_{(g)}{}^{(2)\,\mu}{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta X_{(g)}{}^{(2)\,\theta}{}_\theta & \delta X_{(g)}{}^{(2)\,\theta}{}_\phi \\ 0 & 0 & \frac{\delta X_{(g)}{}^{(2)\,\theta}{}_\phi}{\sin^2\theta} & \delta X_{(g)}{}^{(2)\,\phi}{}_\phi \end{pmatrix}.$ **yields**

$\Longrightarrow$ $\delta X_{(g)}{}^{(2)\,\mu}{}_\nu = 0.$

From the relation $\delta X_{(f)}{}^{(2)\,\mu}{}_\nu = -\frac{1}{\kappa^2 A^2 |ad - bc|}\delta X_{(g)}{}^{(2)\,\mu}{}_\nu,$ $\delta X_{(f)}{}^{(2)\,\mu}{}_\nu = 0.$

$\delta g^{(2)}{}_{pq}$ & $\delta f^{(2)}{}_{pq}$ are related through

$$\delta\gamma^{(2)\,p}{}_q = -\frac{1}{A^2 - A(a+d) + ad - bc}\delta\gamma^a{}_P(\bar{\gamma}^P{}_Q - (\bar{\gamma}^S{}_S - A)\delta^P{}_Q)\delta\gamma^Q{}_q$$

$$(p, q = \theta, \phi, \qquad P, Q, S = t, r)$$

Summary of second order perturbations

$$\begin{cases} \delta G[g]^{(2)\mu}{}_{\nu} = \delta T_{(g)}{}^{(2)\,\mu}{}_{\nu}, \\ \delta G[f]^{(2)\mu}{}_{\nu} = \delta T_{(f)}{}^{(2)\,\mu}{}_{\nu}, \\ \gamma^{(2)p}{}_{q} = -\frac{1}{A^2 - A(a+d) + ad - bc}\delta\gamma^{p}{}_{P}(\bar{\gamma}^{P}{}_{Q} - (\bar{\gamma}^{S}{}_{S} - A)\delta^{P}{}_{Q})\delta\gamma^{Q}{}_{q}. \end{cases}$$

Even if we go into second order perturbations,
the equations of motion for the perturbations of the two metrics
coincide with the perturbed Einstein equations,
though $\delta g^{(2)}{}_{pq}$ and $\delta f^{(2)}{}_{pq}$ are related.

Gauge symmetry representing choice of coordinate

Graviton degrees of freedom : 10x2 - 4x2 – 3 - (4+1) = 4

(Hamiltonian & momentum constraints) **Additional gauge symmetry**

Coincides with those of two massless gravitons.

Summary

● We have investigated the perturbations of a class of spherically symmetric solutions ($\alpha_4 = 1 + \alpha_3 + \alpha_3^2$) in massive gravity and bi-gravity, for which the background EOMs are identical to a set of the Einstein equations with C.C.

● We have found that the interaction terms X in the EOMs for both metrics vanish thanks to the Bianchi identities, and hence the EOMs reduce to the perturbed Einstein equations with the relation between g_{pq} & f_{pq}.

●This feature holds true even for second order perturbations.

●Thus, one cannot distinguish this class of solutions in massive gravity and bi-gravity from the corresponding solutions of GR at least up to second order.

●This fact, however, implies that this class of solutions do not suffer from the non-linear instabilities, which often appear in other cosmological solutions in massive gravity and bi-gravity.

Future work

- **What happens if we go into third order, or, even higher ?**
  Does our argument apply for fully non-linear orders ?
  Is there any hidden symmetry to guarantee the stability ?

- **Does our result still hold if we relax the conditions on background solutions ?**

  Relax spherical symmetry and/or
  the correction term to be (exact) cosmological constant.

- **Does our result still hold if we change matter couplings ?**