Fourier Transform basics

Developing an intuitive understanding of FTs

The Fourier Transform is your friend

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Lectures note for CMB Instrumentation Summer School
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References

There are a lot of good books on Fourier Transforms. Everyone has their favorites.

E.g., mine is Ronald Bracewell, “The Fourier Transform and its Applications”.   It is a old book from my graduate school days.

According to reviews, *This text is designed for use in a senior undergraduate or graduate level course in Fourier Transforms. This text differs from many other fourier transform books in its emphasis on applications. Bracewell applies mathematical concepts to the physical world throughout this text, equipping students to think about the world and physics in terms of transforms. The pedagogy in this classic text is excellent. The author has included such tools as the pictorial dictionary of transforms and bibliographic references.*

For a more modern, detailed reference, maybe see
E. Oran Brigham, “The Fast Fourier Transform”

Applications

Fourier Transforms take temporal or spatial) varying signal and transform them to the frequency or wavelength domain.

Many physical phenomena are readily understood in terms of Fourier Transforms.

E.g.,

- Antennas: far field “beam” pattern is the F.T. of the illumination distribution in the aperture plane.
- Optics: F.T. relation exist between the light amplitude distribution at one focal plane of a lens to the other focal plane. A lens is a F.T. device.
- Astronomical Interferometry (ALMA, VLA, etc.): the measured “visibility function” is the F.T. of the sky brightness.
- Quantum Mechanics: **x** and **p** are related by a F.T. (consider the uncertainty principle).
- Spectrometry, Filters, etc are all understood in terms of F.T.

The Fourier Transform pair

We'll use the explicit notation, and just one dimensional

Fourier Transform $$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi f t}\, dt$$

Inverse F. T. $$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{+i2\pi f t}\, df$$

Show inverse is actually the original function:

show $h'(t)$ obtained by taking inverse F.T. $\{H(f)\}$ equals $h(t)$

$$h'(t) = \int_{-\infty}^{\infty} H(f)\, e^{+i2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t')\, e^{-i2\pi f t'}\, dt'\, e^{+i2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty} h(t') \underbrace{\left[\int_{-\infty}^{\infty} e^{-i2\pi f (t-t')}\, df \right]}_{=\delta(t-t')} dt'$$

$$h'(t) = \int_{-\infty}^{\infty} h(t')\, \delta(t-t')\, dt' = h(t) \quad \checkmark$$

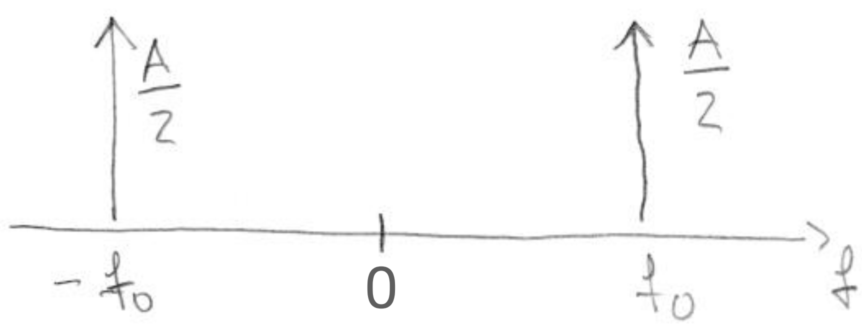
Simple relationships

Some simple, but very helpful, reminders.

What is $F.T.\{A\cos(2\pi f t\} = ?$

$$h(t) = A\cos(2\pi f_0 t)$$

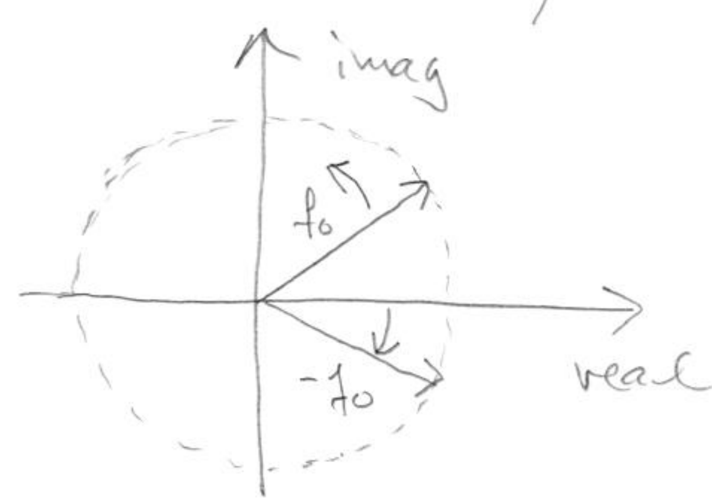
$\Rightarrow H(f)$



signal split between negative and positive signals.

i.e., $A\cos(2\pi f_0 t) = \frac{A}{2}\left(e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}\right)$

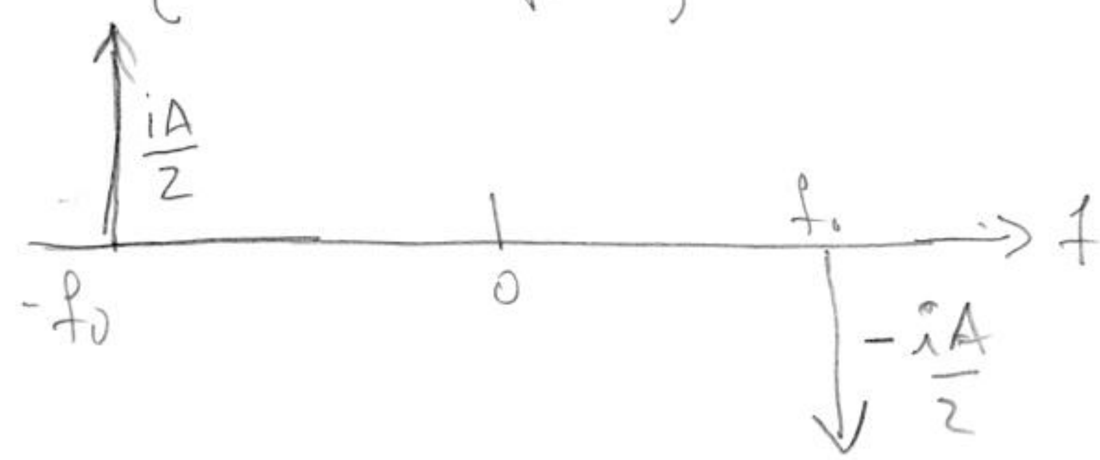
need both +/- f_0 to make it real



Simple relationships

likewise $F.T.\left\{A \sin(2\pi f_0 t)\right\}$

$\Rightarrow$



$$\Rightarrow \frac{-iA}{2}\delta(f-f_0) + \frac{iA}{2}\delta(f+f_0)$$

just keep in mind that $e^{+i2\pi f t} = \cos(2\pi f t) + i \sin(2\pi f t)$

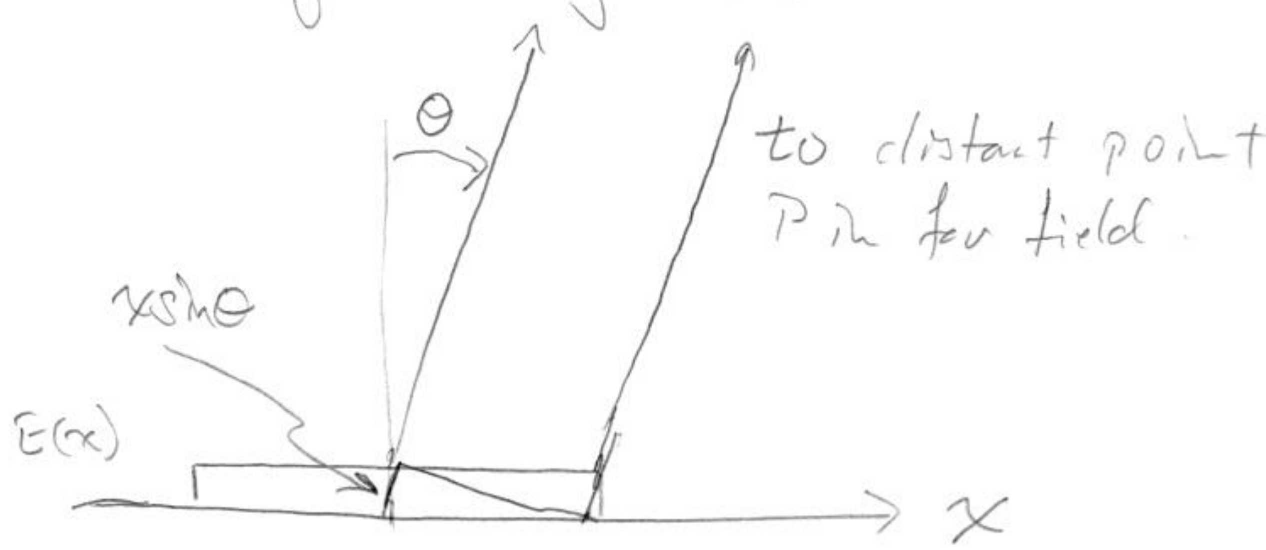
and it's easy to remember
symmetry properties of F.T.

$h(t) \longleftarrow F.T. \longrightarrow \mathcal{H}(f)$

real, even	real, even	Hermitean
real, odd	imag, odd	
imag, even	imag, even	
imag, odd	real, odd	

Example: 1-D aperture and beam

Consider a 1-dimensional aperture with electric field distribution given by $E(x)$



Use Huygen's Principle to obtain the electric field at distance point P as shown. In far field, $R >> D^2/\lambda$, rays are parallel and therefore contribution from region x to $x+dx$ on aperture is

$$E(x)\, e^{-i 2\pi x \sin\theta/\lambda}\, dx$$ ← phase

substitute u for x/λ, redefine E as $E(u)$ and substitute s for $\sin\theta$ (small angle approx.)

$$E(u)\, e^{-i 2\pi u s}\, du$$

Integrate over all space

$$F(s) = \int_{-\infty}^{\infty} E(u)\, e^{-i 2\pi u s}\, du$$

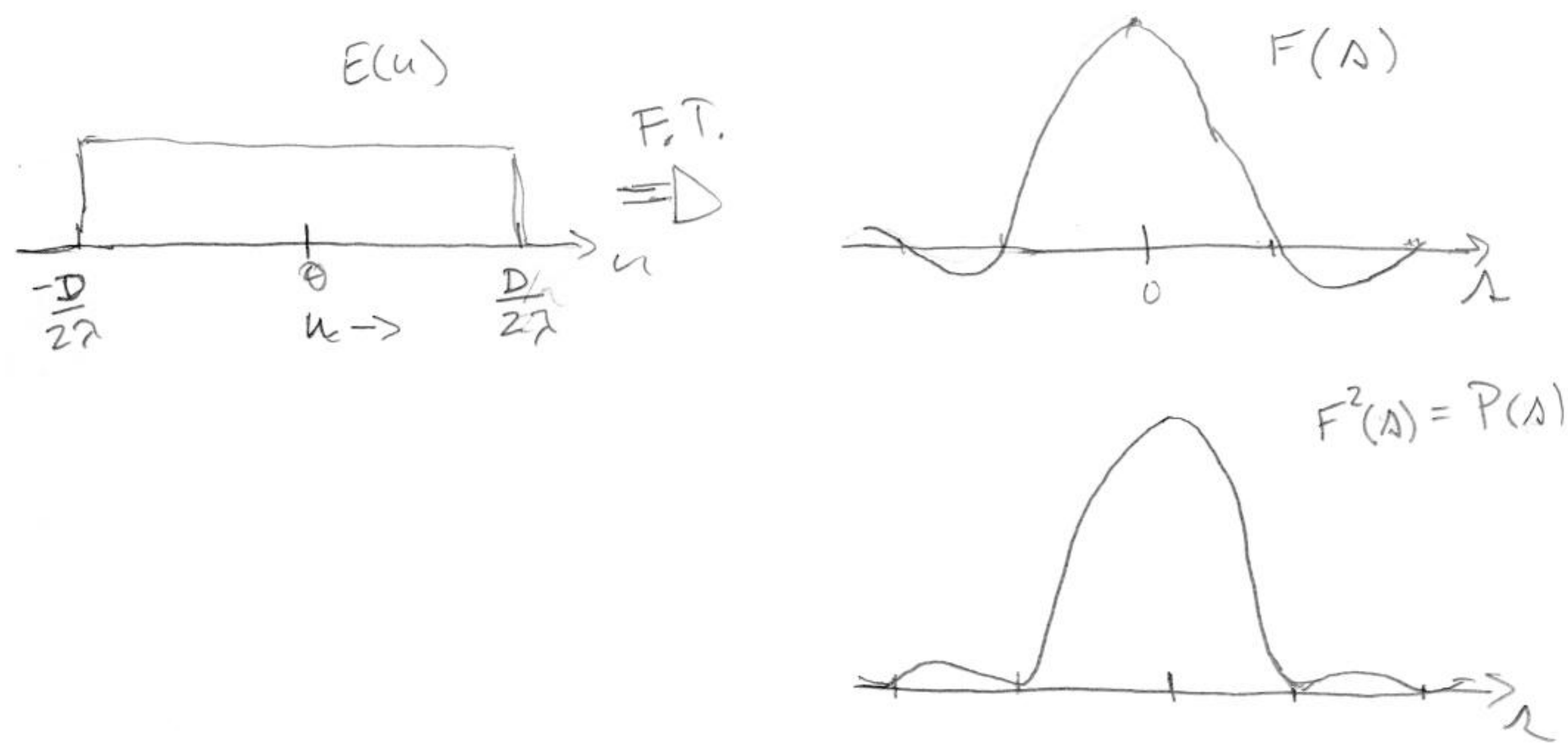
Example: 1-D aperture and beam

F.T. pair: $$F(s) = \int_{-\infty}^{\infty} E(u)\, e^{-i2\pi u s}\, du$$

$$E(u) = \int_{-\infty}^{\infty} F(s)\, e^{i2\pi u s}$$

F(s) is <u>field pattern</u>, not a power response pattern.

To obtain power response pattern you square F(s)

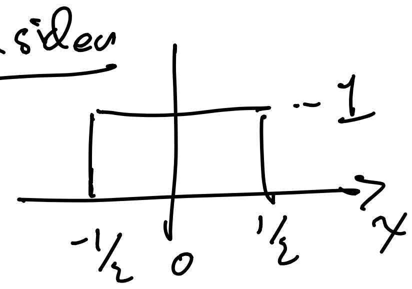


Aside: never forget the Sinc function

(it is always there)

The sinc function is extremely important and comes up all the time in signal processing, imaging, medicine etc. Anytime when one pixelates or samples.

consider



$f(x) = 1$ for $|x| \leq \frac{1}{2}$

Take F.T. $= \int_{-1/2}^{1/2} e^{-i 2\pi f x}\, dx$

clearly only picks out cosine component in F.T.

$$= \int_{-1/2}^{1/2} \cos(2\pi f x)\, dx$$

$$= \frac{1}{2\pi f} \sin(2\pi f x) \Big|_{-1/2}^{1/2}$$

$$= \frac{\sin(\pi f)}{\pi f}$$

Aside: never forget the Sinc function

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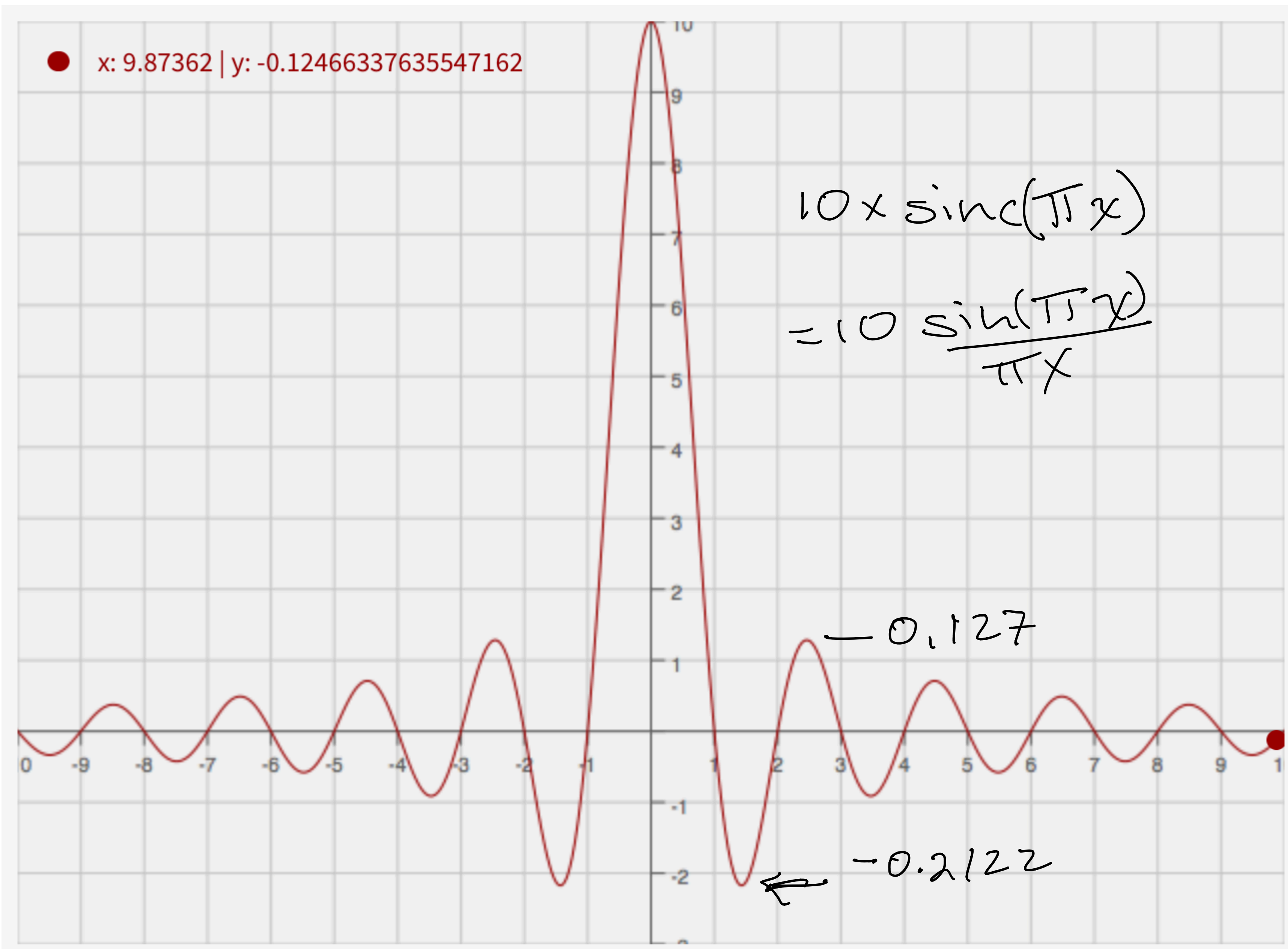

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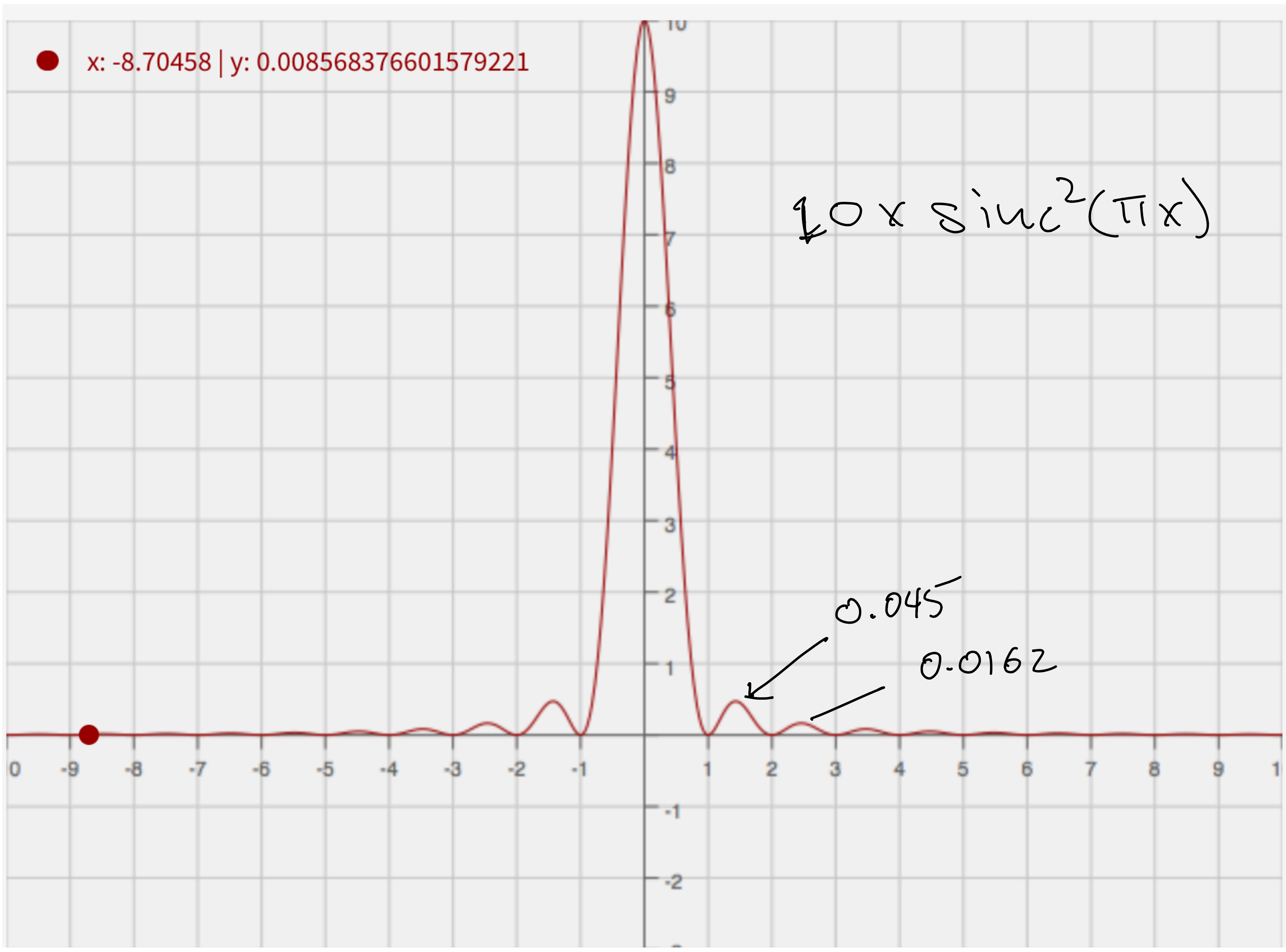
Impact on power spectrum, i.e., squared



Convolution and the Convolution Theorem

Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$y(t) = x(t) * h(t)$$

Symmetric $\quad x(t) * h(t) = h(t) * x(t)$

Associative $\quad f(t) * [x(t) * h(t)] = [f(t) * x(t)] * h(t)$

Distributive $\quad f(t) * [x(t) + h(t)] = f(t) * x(t) + f(t) * h(t)$

Convolution theorem

take $FT\{x(t) * h(t)\}$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-2\pi i f t}\, dt \right]}_{= e^{-i2\pi f \tau} H(f)} d\tau$$

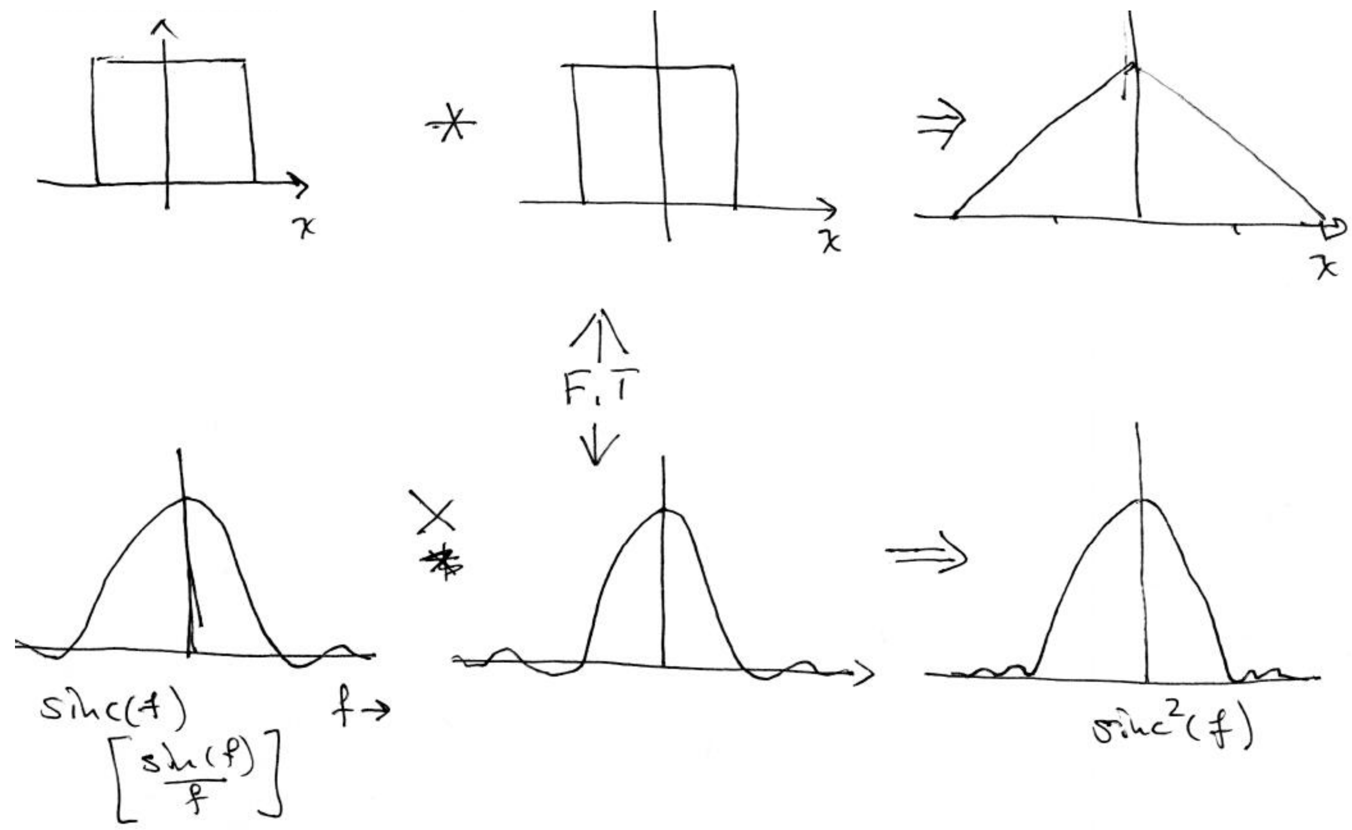
$$= X(f)\, H(f)$$

$$\boxed{F.T.\{x(t) * h(t)\} = F.T.\{x(t)\}\, F.T.\{h(t)\}}$$

This is extremely useful!

Convolution and the Convolution Theorem

Example:



Filters

Linear Systems

Standard signal processing devices are usually linear systems. For example spectrometers, gratings, cameras, CCD's, interferometers, etc are linear systems. We avoid non-linear regime of devices, i.e., saturation.

Our system may include inherently "non-linear" devices, such as multipliers etc., but we mean linear in the sense

$$y(t) = H[x(t)]$$ where H is linear operator

linearity means $y_1(t) + y_2(t) = H[x_1(t) + x_2(t)]$

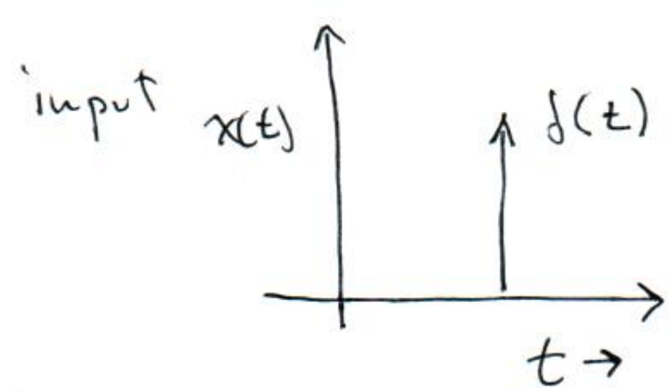
Filters

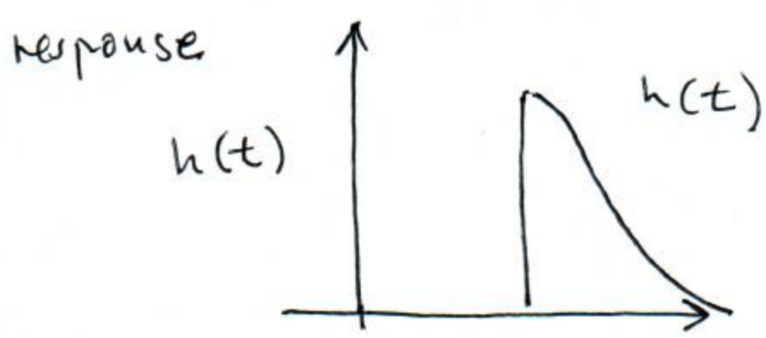
Impulse Response Function

All linear systems can be characterized by an impulse response function, $h(t)$

$$h(t) \equiv H[\delta(t)]$$

response to a δ function input





We can consider the input to be a summation of δ-functions (linear) then

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$y = x * h$$ ← Convolution

Filters

$$y = x * h$$ ← Convolution

This, again, is simplified in Fourier Domain and convolution theorem $F.T.\{y(t)\} = F.T.\{x(t) * h(t)\}$

$$Y(f) = X(f)\,H(f)$$ where CAPS ⇒ F.T.{ }

⇒ all linear systems can be considered filters in Fourier domain ⇐

Applies to time and other domains, of course.

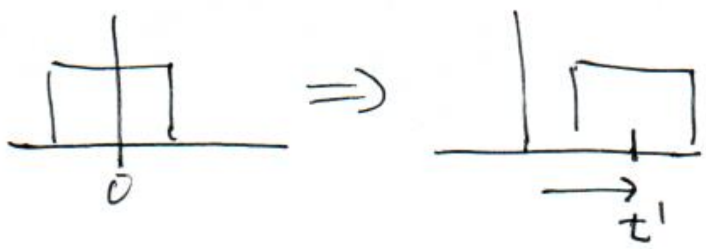
Example: Consider making a map of a point source (unresolved object, like a star) with a telescope. The point source will appear in your map with the pattern of the "Point Spread Function" (PSF). The PSF is the impulse response of your telescope. The map you produce is the **convolution of real sky image and the PSF.**

Your telescope acts like a filter that cannot sample "spatial frequencies" larger than its diameter to be measured. The result is the image is resolution is limited to observing wavelength / telescope diameter, λ/D.

A F.T. of your map will be bandwidth limited.

Handy Theorems

Shift Theorem: What happens to transform if function is shifted in time?



$$f(t) \Rightarrow f(t-t')$$

$$F(f) \Rightarrow F(f)e^{-i2\pi f t'} \leftarrow \text{phase gradient}$$

$$\boxed{\text{Shift Theorem}}$$

Parseval's Theorem or Rayleigh's Theorem or Energy Theorem:

Parseval's Theorem or Rayleigh's Theorem or "energy theorem"

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df$$

makes sense – energy in one domain equals that in the other

Another obvious and very useful relationship:

$$\int_{-\infty}^{\infty} h(t)dt = H(0) \leftarrow \text{"DC" component}$$

(zero frequency)
(from DC ~~✓~~, AC)

and $\int_{-\infty}^{\infty} H(f)\,df = h(0)$

Show Parseval's Theorem

exercise left to the reader

show it

$$\int_{-\infty}^{\infty}|h(t)|^2 dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} H(f)\, e^{i2\pi f t} df \int_{-\infty}^{\infty} H^*(f')\, e^{-i2\pi f' t} df' dt\right.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f) H^*(f') \underbrace{\int_{-\infty}^{\infty} e^{i2\pi(f-f')t} dt}_{\delta(f-f')} \, df df'$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 df \quad \checkmark$$

note used $h(t) = \int_{-\infty}^{\infty} H(f)\, e^{i2\pi f t} dt$

and $h^*(t) = \int_{-\infty}^{\infty} H^*(f)\, e^{-i2\pi f t} dt$

$$= \int_{-\infty}^{\infty} H^*(-f)\, e^{i2\pi f t} dt$$

so if $h(t)$ is real the $H(f) = H^*(-f)$

and ~~H~~ is Hermitian

Correlation

<u>Correlation</u>

cross correlation of $x(t)$ with $h(t)$
defined by

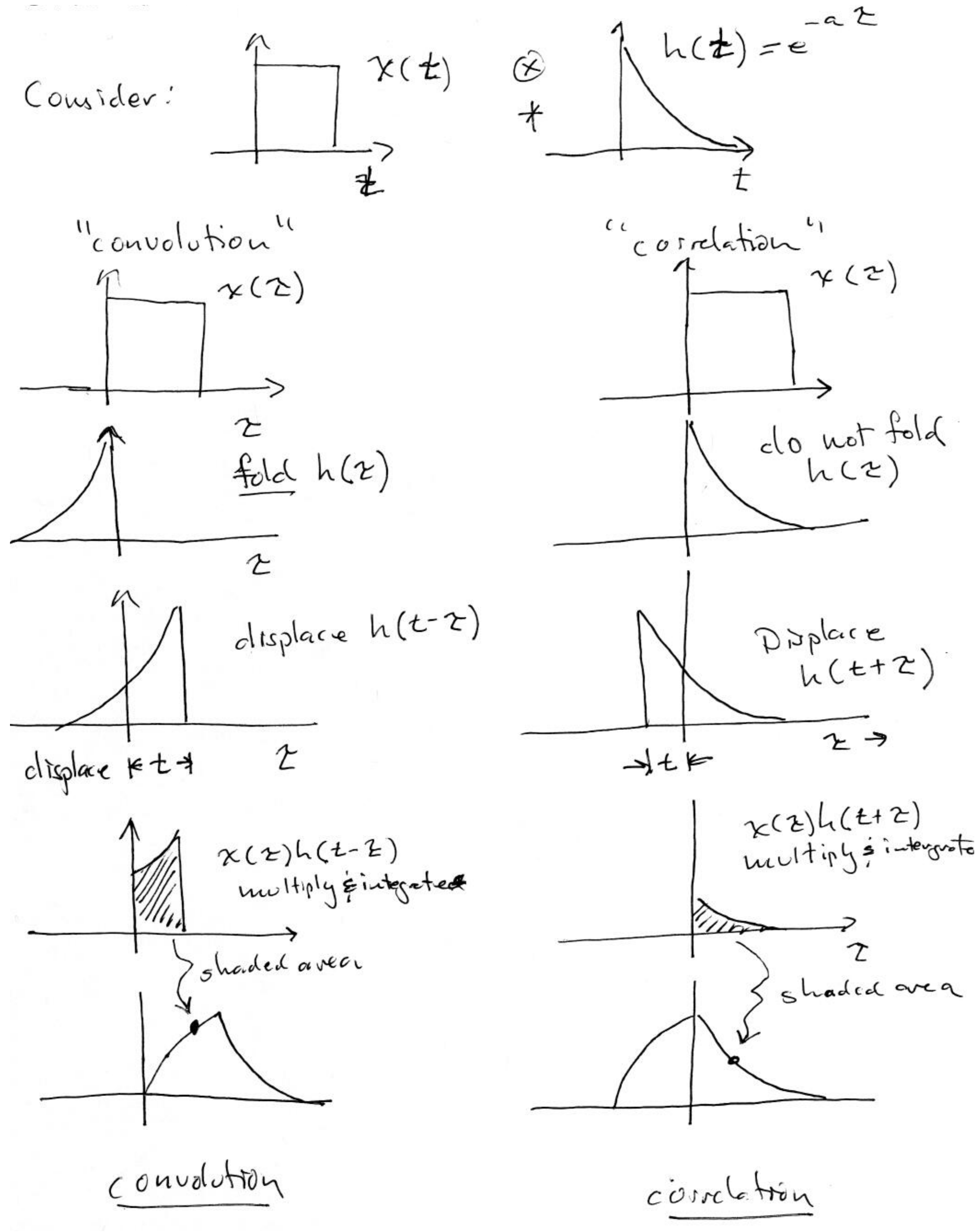
$$z(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t+\tau)\,d\tau$$

$$= x(t) \otimes h(t)$$

Similar to convolution but h is not folded

Correlation compared to Convolution

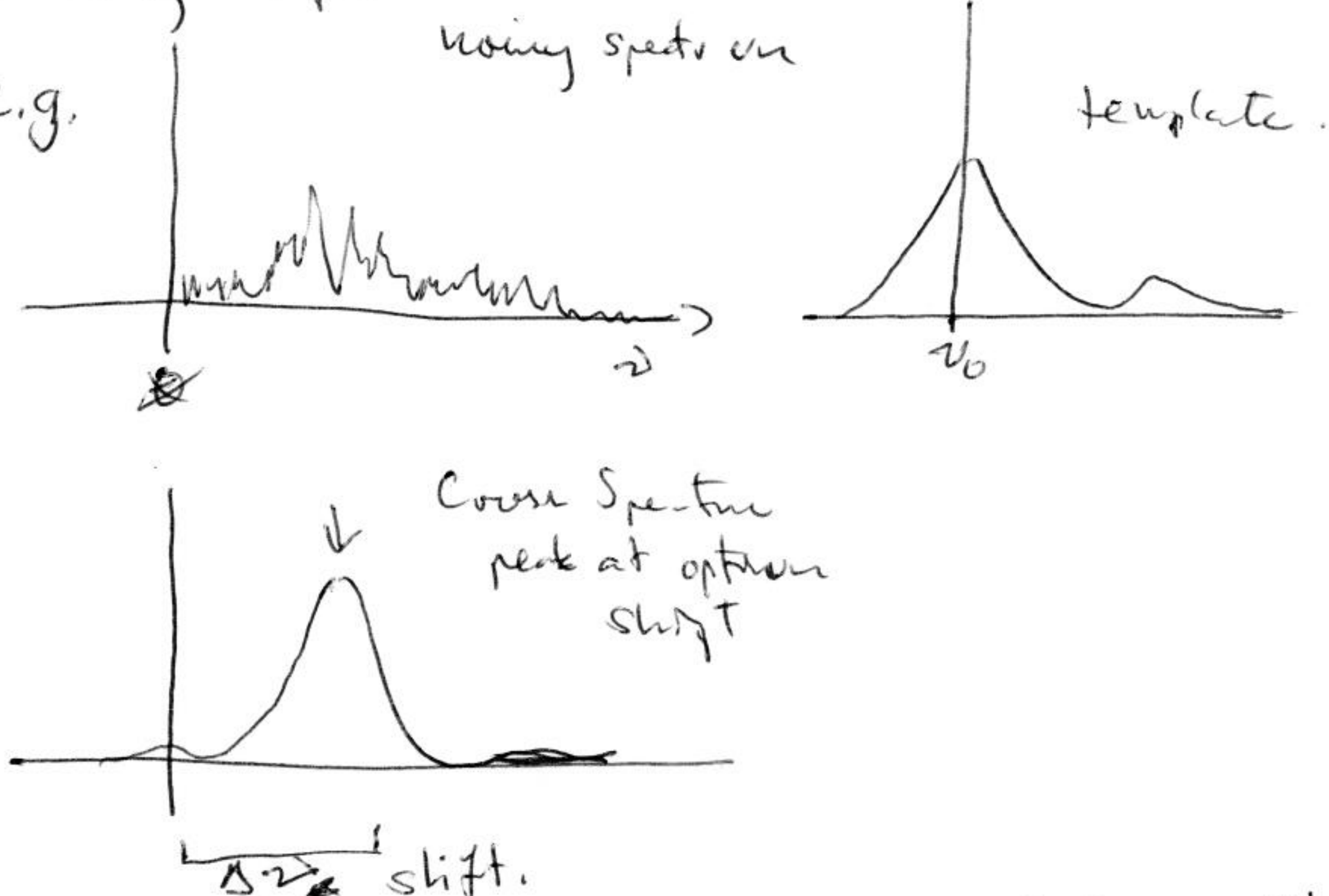
Illustration of Correlation compared to Convolution:



Applications of Correlation

Many applications of correlation, or cross-correlation, such as
- interferometry
- spectroscopy
- determining shifts, e.g.,

Determining shifts



makes it easy to see why if $z(\nu) = g \otimes h$
then $z(-\nu) = h \otimes g$

h(τ) is not folded in correlation, so if z(x) = g⊗h then z(-x) = h⊗g

$$z(x) = \int_{-\infty}^{\infty} g(u)h(x+u)\,du \qquad \text{let } u' = x+u, \quad du' = du, \quad u = u' - x$$

$$= \int_{-\infty}^{\infty} g(-x+u')h(u')\,du'$$

Correlation Theorem, Autocorrelation

Correlation Theorem

$$F.T.\{h \otimes x\} = F.T.\left\{\int_{-\infty}^{\infty} h^*(t)\, x(t+\tau)\, dt\right\} = H^*(f)\, X(f)$$

$\boxed{\text{"cross-correlation" spectrum}}$

Autocorrelation and the Wiener -Khinchin Theorem
- the basis of radio astronomy spectroscopy and Fourier Transform Spectrometers (FTS)

$$C(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

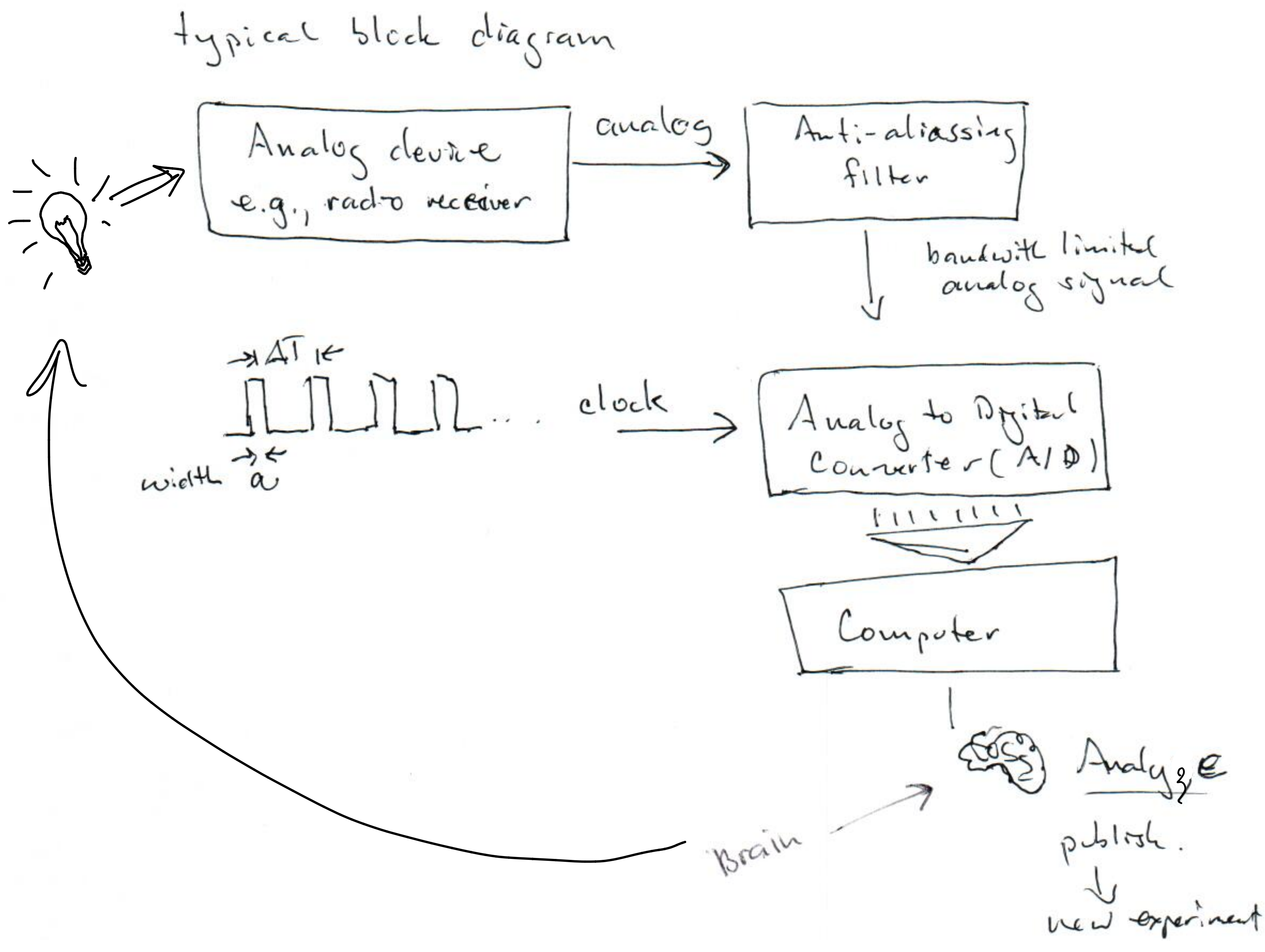
symmetric for x real $\Rightarrow C(\tau) = C(-\tau)$

$$F.T.\{C(\tau)\} = |X(f)|^2$$

$\boxed{\text{power spectrum of } x(t)}$

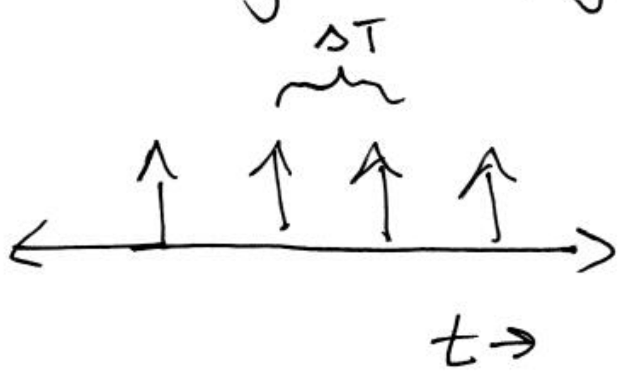
"Wiener-Khinchin Theorem"

Sampling, Sampling Theorem, Nyquist rate



Sampling, Sampling Theorem, Nyquist rate

Given $x(t)$ we convert to sampled data by multiplying by



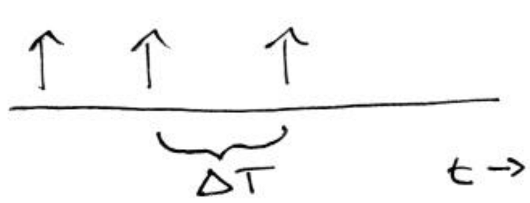
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \equiv \mathrm{III}(t/\Delta T)$$

"Shah Function"

result in $x_s(n) = x(t)\delta(t - n\Delta T)$

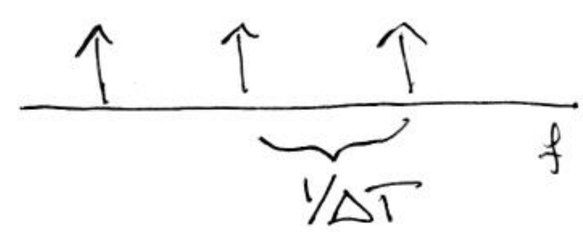
we'll need $\text{F.T.}\{\mathrm{III}(t/\Delta T)\} = \frac{1}{\Delta T}\mathrm{III}(f\Delta T)$

$\mathrm{III}(t/\Delta T)$



$\overset{\text{F.T.}}{\Longrightarrow}$

$\frac{1}{\Delta T}\mathrm{III}(f\Delta T)$



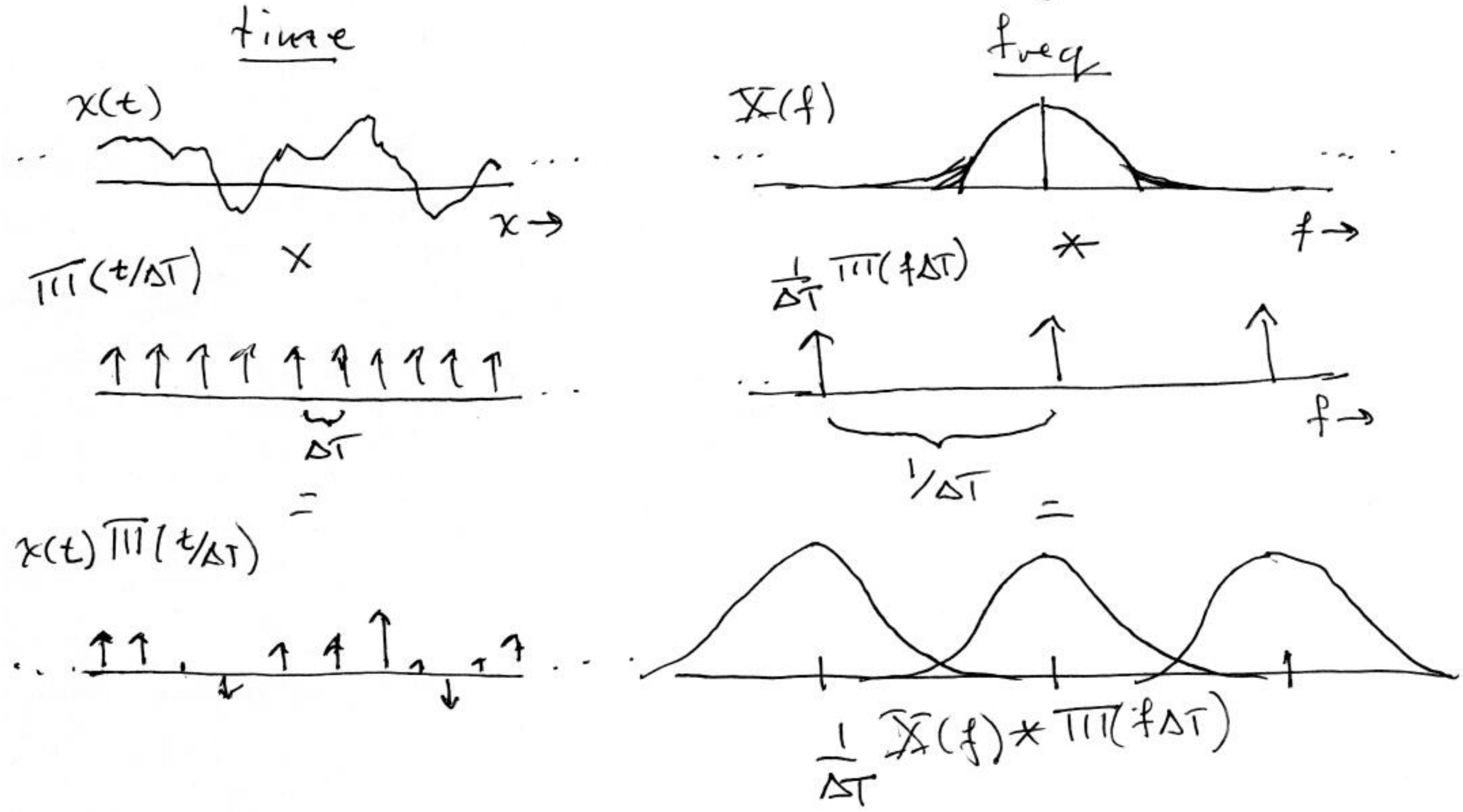
exercise left to the reader

To derive $\text{F.T.}\{\mathrm{III}(t/\Delta T)\}$, consider F.T. of Fourier series expansion of $s(t) = \sum_{n=-\infty}^{\infty} s_n e^{-i2\pi n F_s t}$

with $s_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s(t) e^{-i2\pi n F_s t}\, dt$

Sampling, Sampling Theorem, Nyquist rate

Consider the effect of discrete sampling



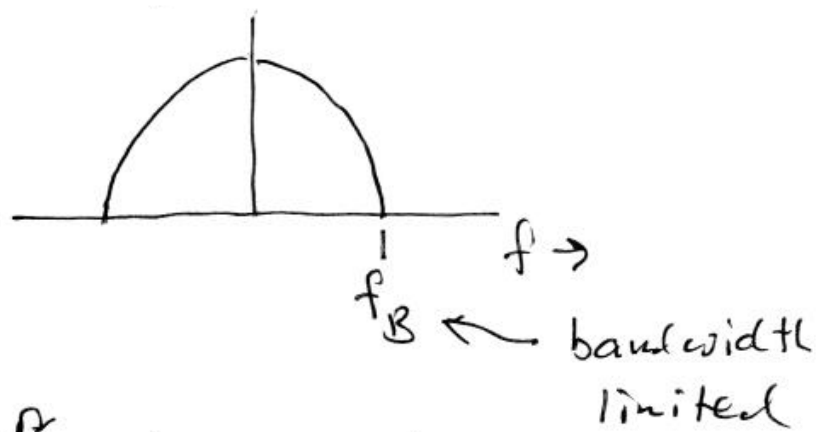
So, sampling has caused the Fourier Transform to be repeated indefinitely with spacing $1/\Delta T$,

For bandwidth limited signal, the minimum sampling rate for which the F.T. of the signal does not overlap is the Nyquist frequency

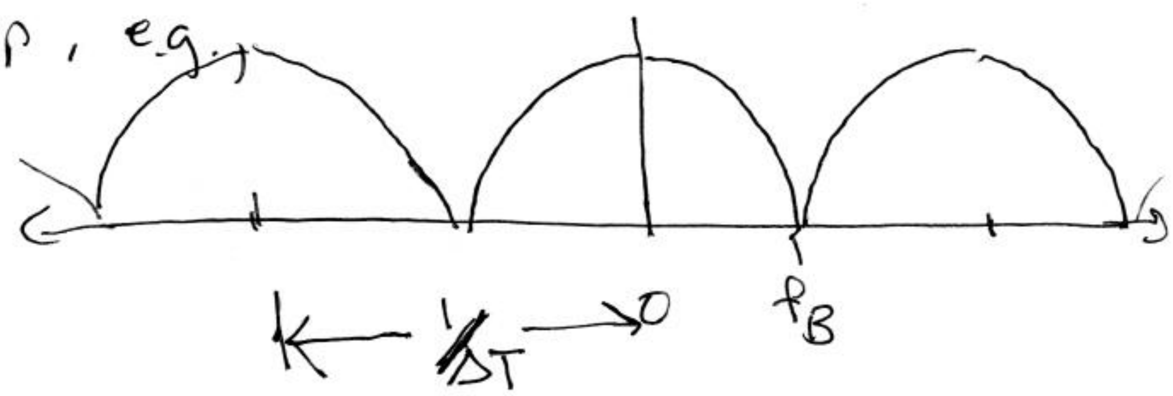
Sampling, Sampling Theorem, Nyquist rate

For bandwidth limited signal, the minimum sampling rate for which the F.T. of the signal does not overlap is the Nyquist frequency

If $X(f) \Rightarrow$



then need $\frac{1}{\Delta T} \geq 2 f_B$ to avoid overlap, e.g.,



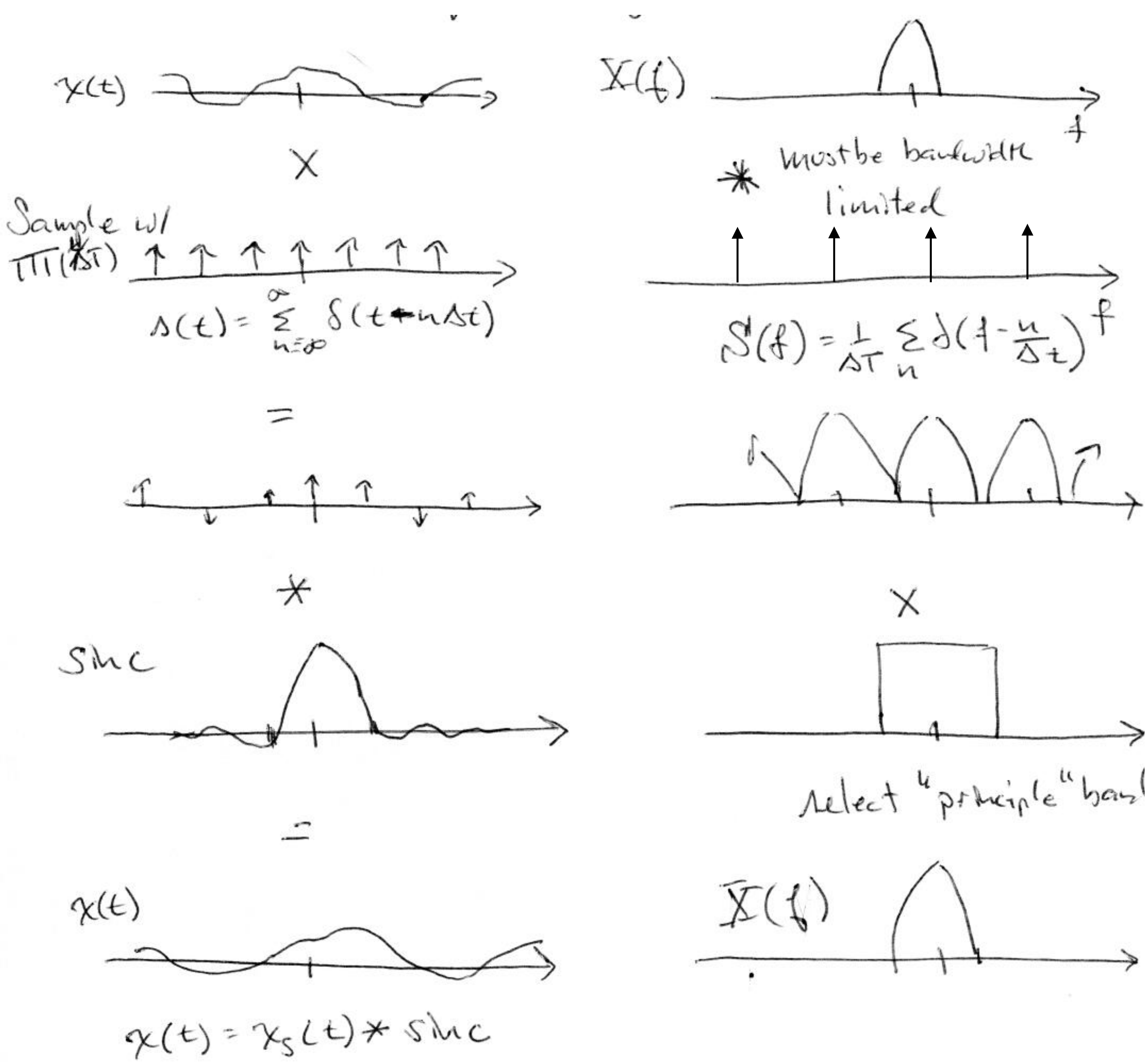
$$\boxed{\text{Nyquist frequency} = 2 f_B}$$

Sampling, Sampling Theorem, Nyquist rate

Given that no information is lost if we sample at rate greater than the Nyquist frequency, how do we reconstruct the original x(t) from the samples $x_s(k)$? This is given by the sampling theorem:

$$x(t) = \sum_{k=-\infty}^{\infty} x_s(k)\,\mathrm{sinc}\left[2\pi f_B (t - k\Delta T)\right]$$

Graphical proof of the sampling theorem

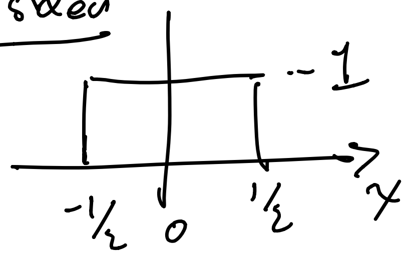


time domaine *frequency domaine*

Aside: never forget the Sinc function

(it is always there)

consider



$$f(x) = 1 \text{ for } |x| \le 1/2$$

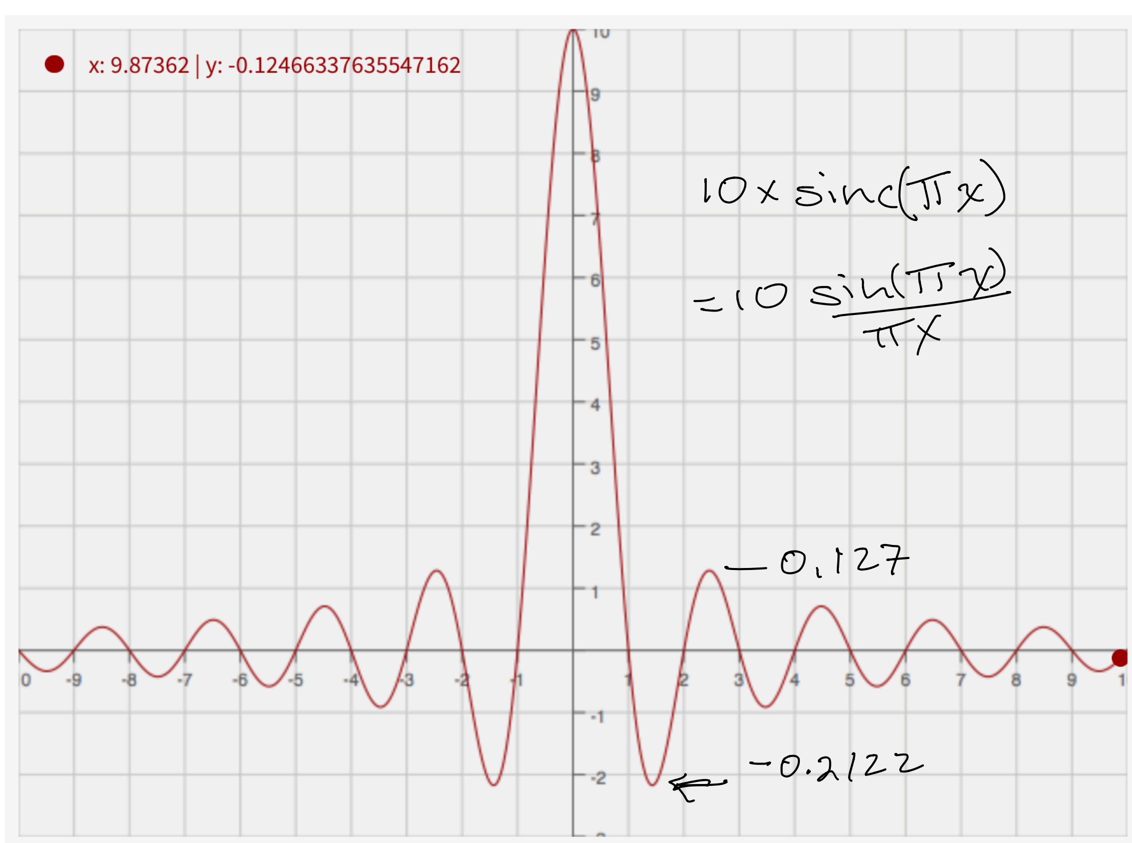
Take F.T. $= \int_{-1/2}^{1/2} e^{-i2\pi f x}\, dx$

clearly only picks out cosine component in F.T.

$$= \int_{-1/2}^{1/2} \cos(2\pi f x)\, dx$$

$$= \frac{1}{2\pi f} \sin(2\pi f x) \Big|_{-1/2}^{1/2}$$

$$= \frac{\sin(\pi f)}{\pi f}$$



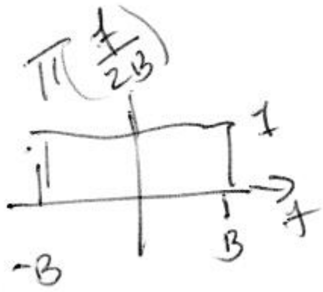
Sampling, Sampling Theorem, Nyquist rate

exercise left to the reader

Write out "proof" of sampling function

We know $\quad F.T.\{x(t)\} = X(f)$

$$F.T.\{x_s(t)\} = X(f) * \frac{1}{\Delta T} Ш\left(\frac{f}{\Delta T}\right)$$



multiply by boxcar function of width 2B and height=1

e.g., $\Pi\left(\frac{f}{2B}\right)$ where $2B = \frac{1}{\Delta T}$; B is bandwidth

This gives principle band, which is true $F.T.\{x(t)\}$ with inverse $F.T. = x(t)$

$$x(t) = \text{inverse } F.T. \left\{ \left[X(f) * 2B\, Ш\left(\frac{f}{2B}\right)\right] * \Pi\left(\frac{f}{2B}\right) \right\}$$

I.F.T.

$$= \sum_{n=-\infty}^{\infty} \underbrace{x_s(n \cdot \Delta T)}_{\text{samples}} \underbrace{\operatorname{sinc}\left[2\pi B(t - n\Delta T)\right]}_{\text{convolved w/ sinc}}$$

Works as long as $\frac{1}{\Delta T} \geq 2B$, where B is bandwidth !!!

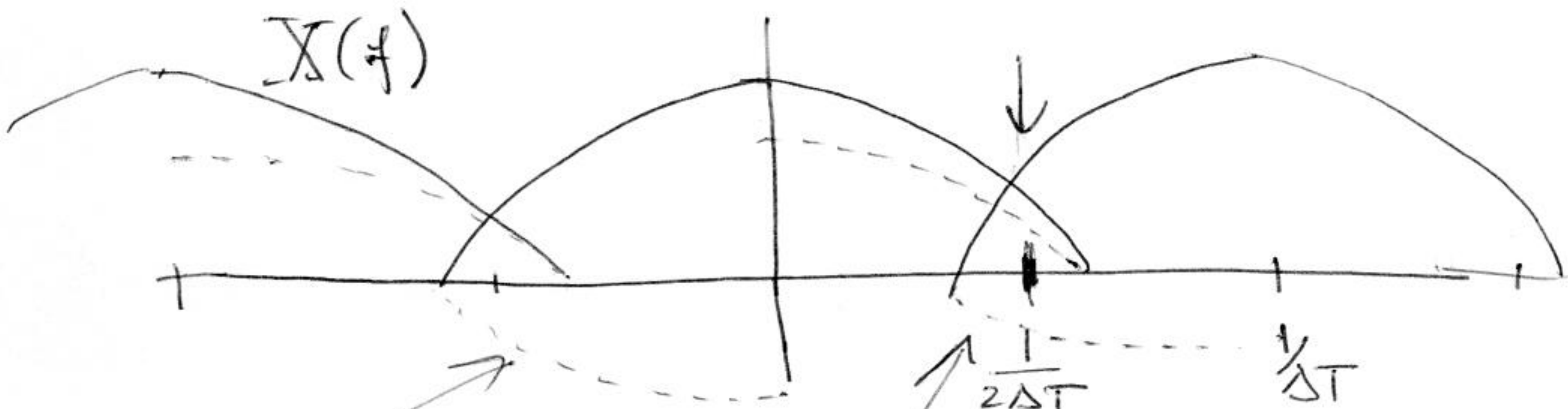
Sampling, Sampling Theorem, Nyquist rate

Comments on sampling

Sampling theorem requires:

1) bandwidth limited signal or information is lost.
2) Bandwidth limited signal means the summation must be over <u>all</u> time! If a signal is bandwidth limited then it cannot be time limited
3) Amplitude of each sample must be absolutely precise.

Aliassing (sometimes called folding)



imag part (odd if x real)

Overlap is referred to as aliassing of the signal. Note that it is the negative frequencies of $X(f)$ overlapping with positive.

Discrete Fourier Transform

Discrete Fourier Transform (DFT)

- do not usually have continuous stream of data, we have samples

- do not have infinite series of samples, N

- use computers, so need a formulation for discrete time (or spatial..) and <u>discrete frequency</u> components.

$\Rightarrow$ we use discrete Fourier transform pair

$$\boxed{\begin{aligned} X(l) &= \sum_{k=0}^{N-1} x(k) \exp\left[-i 2\pi l \Delta f \cdot k \Delta T\right] \\ x(k) &= \frac{1}{N} \sum_{l=0}^{N-1} X(l) \exp\left[+i 2\pi l \Delta f \cdot k \Delta T\right] \end{aligned}}$$

- where k, l are indices for time & freq. bins, respectively

- usually $\Delta f \Delta T = \frac{1}{N}$

- Note, asymmetric $\rightarrow \frac{1}{N}$ in inverse transform

Discrete Fourier Transform

- Need to decide values of Δf, ΔT, N to use
- Important to understand differences of DFT from continuous Fourier Transform.

$\Rightarrow$ Helpful for understanding to "derive" DFT from operations on continuous functions using Ш "shah" functions.

Note: discrete Fourier Transform pairs are periodic.

This makes it easy to understand apparently odd behaviour

Discrete Fourier Transform

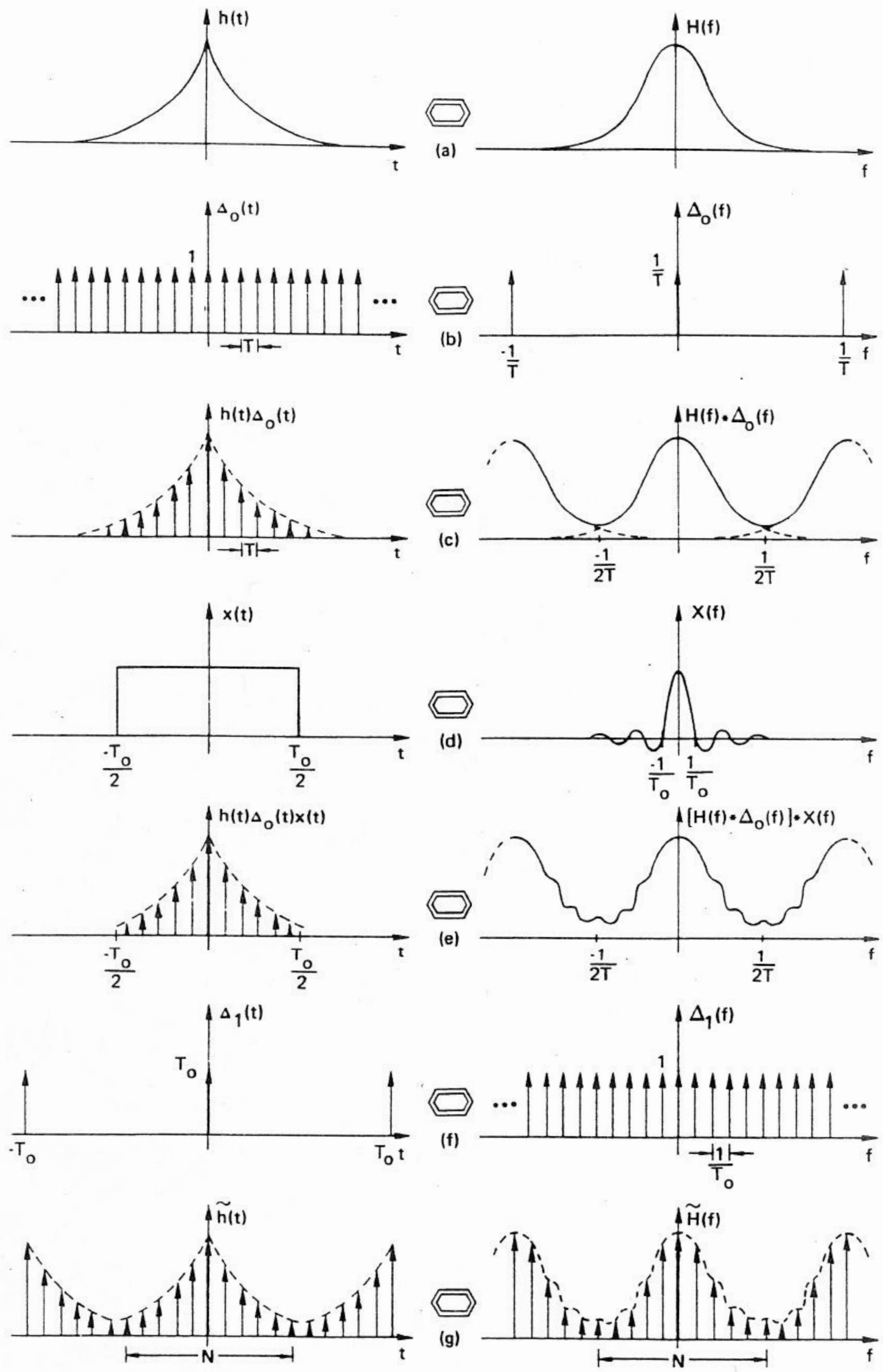


Figure 6-1. Graphical development of the discrete Fourier transform.

Discrete Fourier Transform

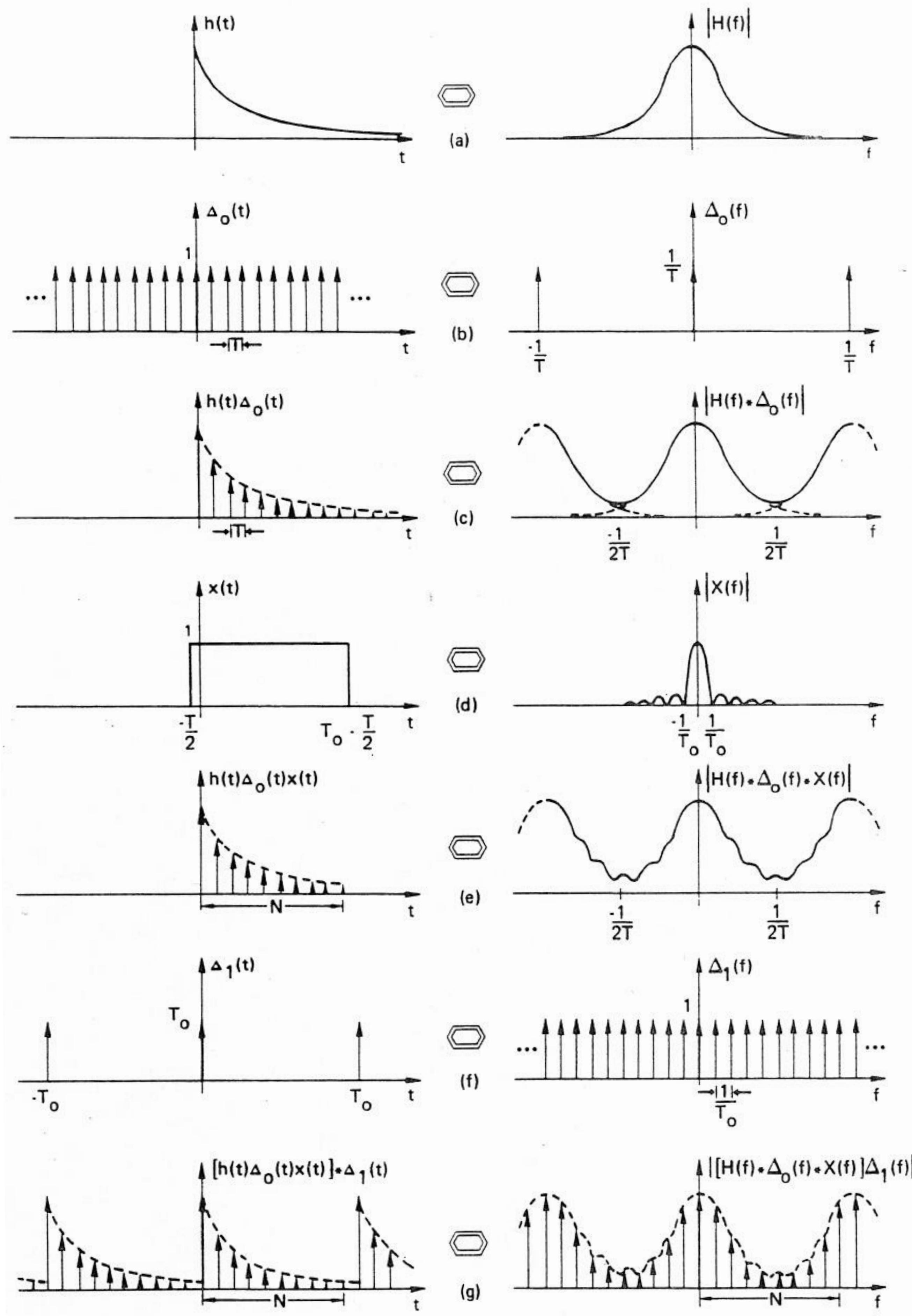
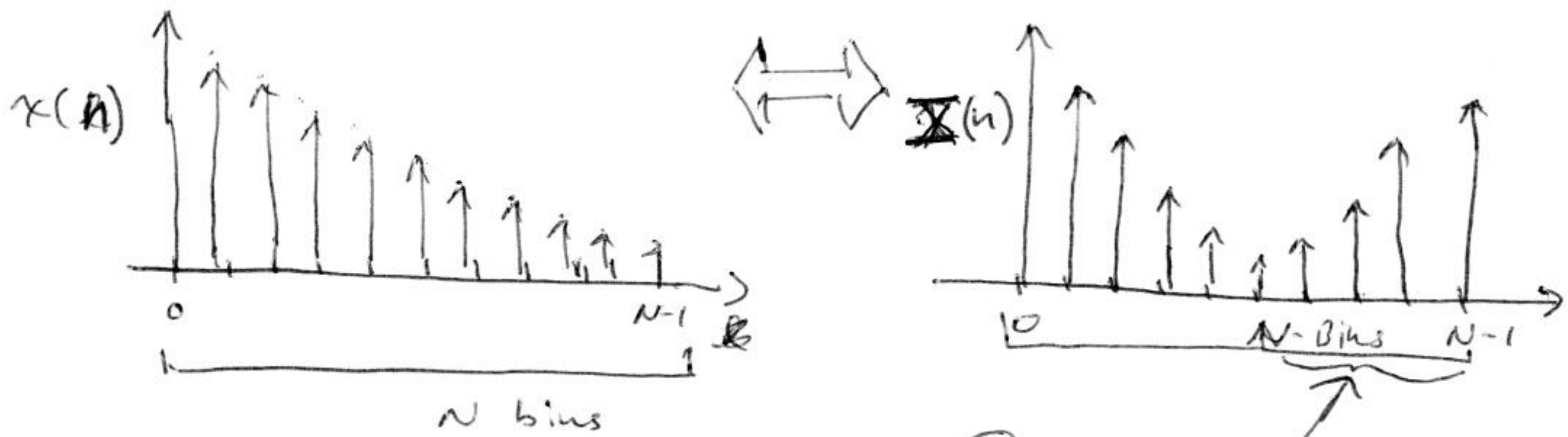


Figure 6-2. Graphical derivation of the discrete Fourier transform pair.

Discrete Fourier Transform

Comments on DFT



① Note: There are the negative freq. components!
Be careful when interpreting DFT'S.
(read the fine print)

② Note: in the development of the DFT, you multiplied by boxcar function in the time domain
So, you have convolved the F.T. with sinc function. This leads to distortions ("mode mixing") in the DFT and effects the frequency resolution.

Consider example on next page.

There are two equivalent ways to look at this

1) in terms of the window function (next section)

2) from the discontinuity caused by the replication of the function — the discontinuity at the periodic boundaries —
→ more frequency components are needed to produce discontinuity

Discrete Fourier Transform

example from discrete F.T. pair

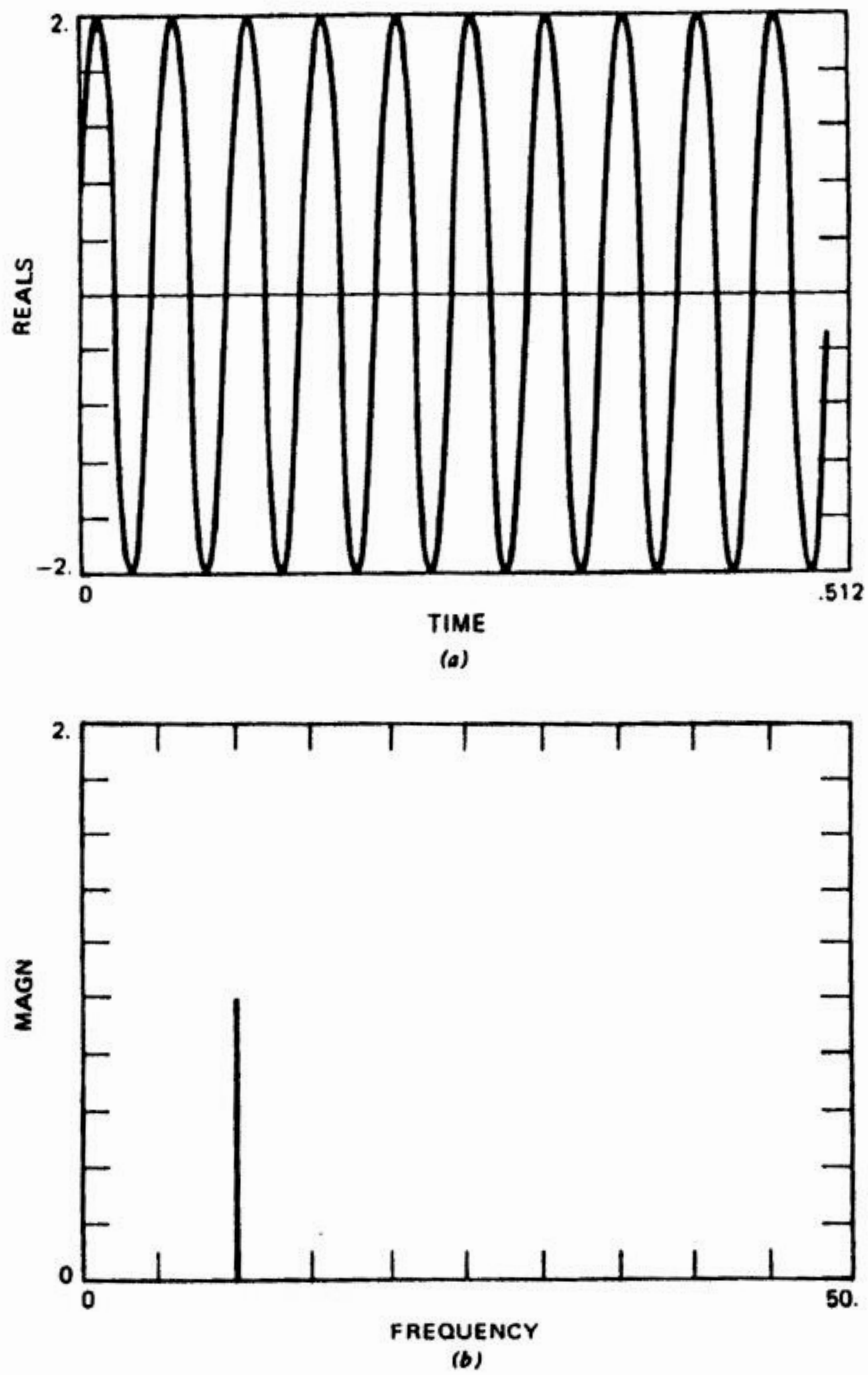
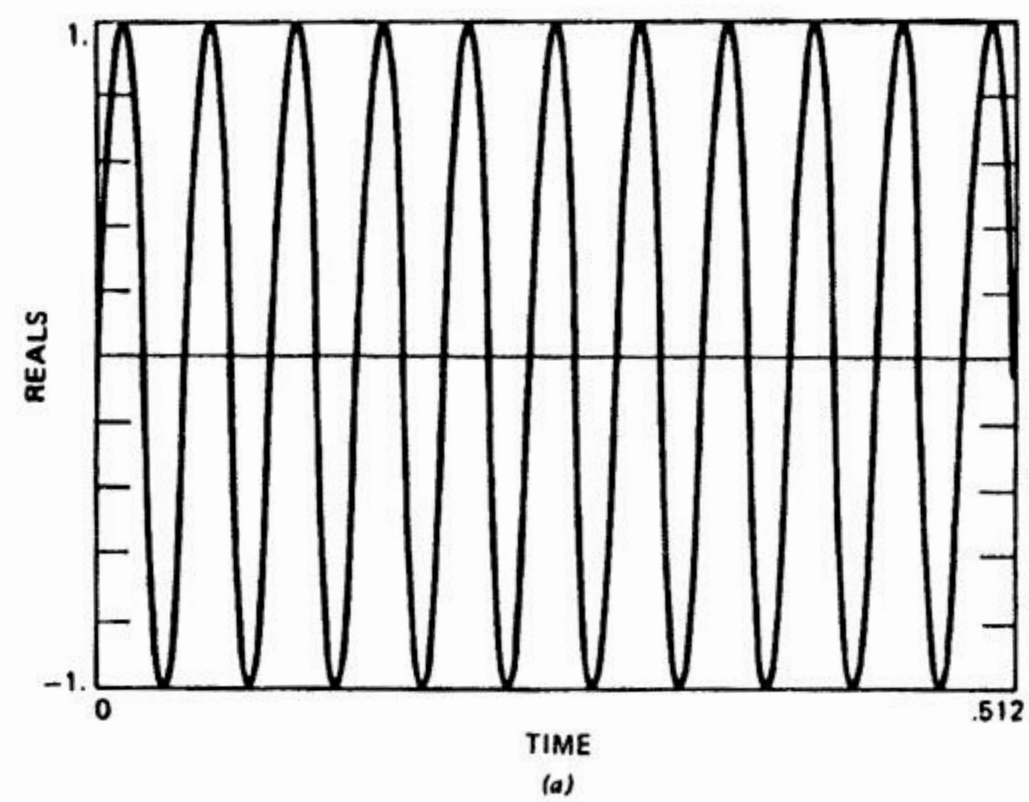


Figure 8.7 PSD of sine wave with integral number of cycles (10 cycles).

Discrete Fourier Transform

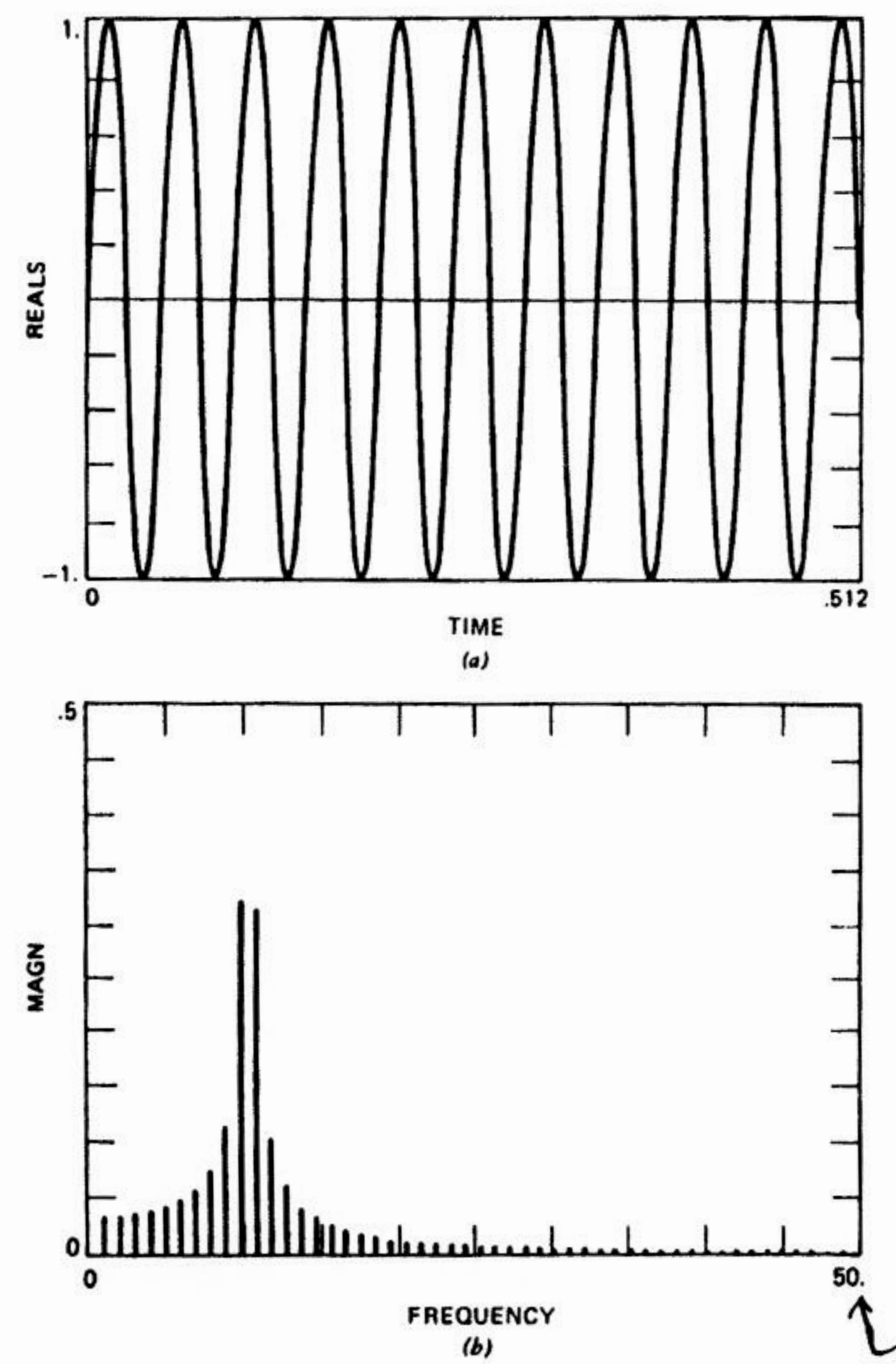
example from discrete F.T. pair



What is the F.T.?

Discrete Fourier Transform

example from discrete F.T. pair



Note, only 50, not 512

Figure 8.8 PSD of sine wave with integral number of cycles plus half cycle (10.5 cycles).

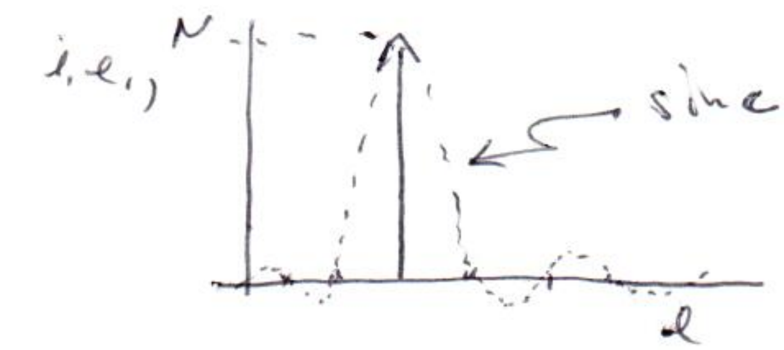
Leakage and window functions

Leakage and window functions

To understand examples on last page (DFT(sin)), remember we have truncated the time series with a boxcar function, and so convolved F.T. with a sinc function.

When the sinusoid is at a multiple of Δf then convolving sinc function is zero at other frequency bins.

i.e.,



but when f midway between bins.

This is referred to as "leakage" or "mode mixing"

NOTE! Parseval's theorem still holds
→ power integrated over spectrum is still preserved.

Window function: Apodization

you can think of the frequency response as "side lobes", (a term borrowed from antenna theory jargon),

We can greatly reduce the sidelobes or leakage by "apodizing" or "tapering" the aperture, but at cost of lower resolution.

Resolution of the uniform window (boxcar fnc) is given by FWHM of sinc,

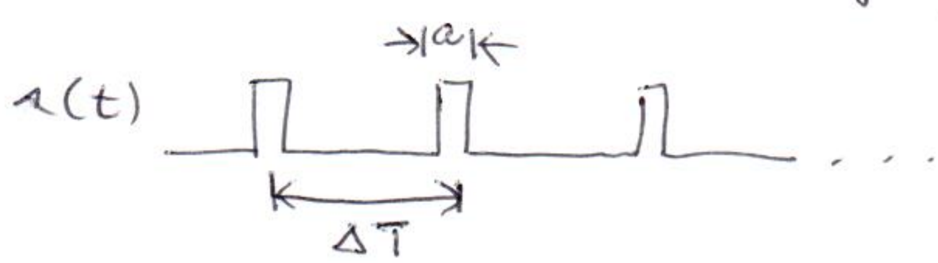
$\rightarrow$ for $\Delta T \Delta f = \frac{1}{N}$, then FWHM(sinc) $= 1.2 \Delta f$

Impact of Averaging or Binning

(it is also always there)

So far we assumed signal sampled by $\delta(t - j\Delta T)$.

What if the "sample and hold" circuit has a finite integration time (averaging time)?



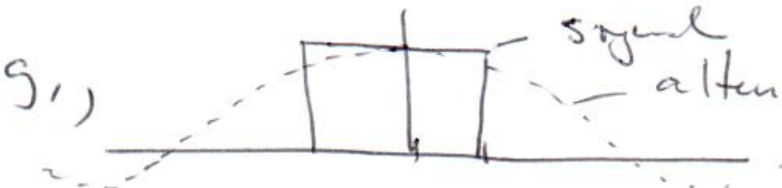
$x_s(t)$ is convolved with boxcar of width a

Results in multiplying the spectrum with F.T. of boxcar $= \mathrm{sinc}(\pi a f)$

$\longrightarrow$ output spectrum $\Rightarrow X(f) \cdot \sin(\pi a f)$

Net result is attenuation of higher frequencies, which can be corrected (but increases noise at high freq.)

e.g.,



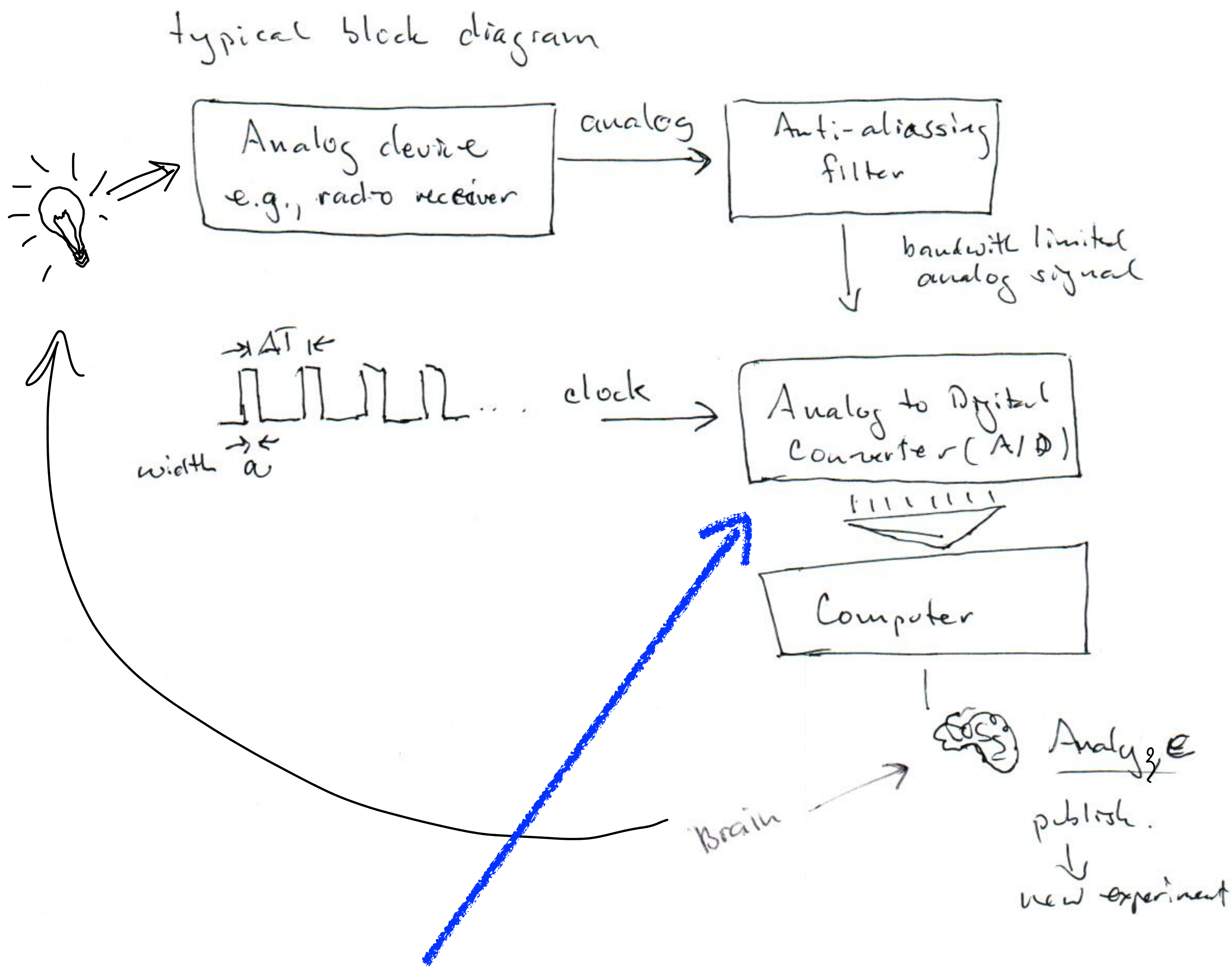
— signal

— attenuation by $\mathrm{sinc}(\pi a f)$

Even if $a = \Delta T$ spectrum at band edge is only attenuated by $\frac{2}{\pi}$

Whatever binning or cell size you use will result in filtering the spectrum.
Easy to correct, but also easy to forget!

Wrap up



Bonus question: What is the impact of digitized samples on the output power spectrum?

Extra material

Impact of digitization

Digitization

$x_s(j)$ - sampled signal (precise value at time $\Delta T j$)

$C_s(j)$ - corresponding counts after quantization "digitization"

e.g.) $x_s(j)$ is a voltage across capacitor, which is converted to $C_s(j)$ using analog to digital converter (A/D or ADC)

Assume ADC converts voltages between $0 \to V_{max}$ to an integer between $0 \to 2^n$ (linear conversion, n bits)

$$C_s(j) = \underbrace{\text{nint}}_{\text{nearest integer}}\left[\alpha \cdot x_s(j)\right]$$

$\alpha = \dfrac{2^n}{V_{max}}$ for an n-bit ADC

Output of ADC is integer, so max error is ± 0.5

$$x_s(j) = \frac{C_s(j)}{\alpha} + \underbrace{\frac{\epsilon(j)}{\alpha}}$$

noise term $|\epsilon(j)| < 0.5$

Impact of digitization

Assume:

1) $\epsilon(j)$ are uniformly distributed over $[-0.5, 0.5]$

then $\sigma_\epsilon^2 = \int_{-1/2}^{1/2} x^2\,dx = \frac{1}{12}$ with $\langle \epsilon \rangle = 0$

2) $\epsilon(j)$ are uncorrelated, $E(\epsilon(j)\epsilon(k)) = \delta_{jk}\sigma_\epsilon^2$

Good assumptions as long as we sample at Nyquist or higher and ADC works well.

What is effect of $\epsilon(j)$ quantization errors on power spectrum?

Compute power spectrum of $\epsilon(j)$

$$a(\ell) = \frac{1}{\alpha}\sum_{k=0}^{N-1} \epsilon(k) \underbrace{\exp\left[-i\,2\pi \ell \Delta f k \Delta T\right]}_{\equiv f(k,\ell)}$$

Fourier components

$$\text{power spectrum} = a(\ell)a^*(\ell) = \frac{1}{\alpha^2}\sum_k\sum_m \epsilon(k)\epsilon^*(m)\, f(k,\ell)\, f^*(m,\ell)$$

and take expectation value

$$E\left[a(\ell)a^*(\ell)\right] = \frac{N\sigma_\epsilon^2}{\alpha^2}$$

Note independent of ℓ, "white" noise just adds noise floor to spectrum

Impact of digitization

limit to dynamic range

dynamic range $\equiv \dfrac{\text{max signal power}}{\text{noise power}}$

consider sinusoid at max amplitude $= \dfrac{V_{max}}{2}$

Power spectrum of signal $\dfrac{N^2 V_{max}^2}{16} \delta(f \pm f_0)$

Dynamic range $\simeq \dfrac{N^2 V_{max}^2}{16} \Big/ \dfrac{1}{\alpha^2} N \sigma^2$

$\simeq \dfrac{3}{4} N \, 2^{2n}$

Consider $n=10$, $N=1024 \Rightarrow$ dynamic range ~ 90dB

(10^9)

in 1^{st} bin.

If integrate over N bins, total dynamic range

$\simeq \dfrac{3}{4} 2^{2n} \quad \sim 6$dB/bit

10 bit ~ 60dB $= (10^6)$

Rule of thumb 6dB/bit