In frequency space, any signal can be represented as \( I \cdot \sin(\omega t) + Q \cdot \cos(\omega t) \).

For linear circuits, we can consider each \( \omega \) independently.

The circuits we'll be discussing look like this:

\[ S_{21} = \text{Amplitude of } \sin \text{ comp. at } P_2 + i \cdot \text{Amplitude of } \cos \text{ comp.} \]

\[ S_{11} = \text{the same, but w.r.t. power reflected from the device back to } P_1 \]
**Resonator Circuit:**

\[ \omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}} \]

\begin{center}
\[ |S_{21}| \]
\end{center}

\[ Q_r = \frac{f_0}{df} = \text{fraction of power lost per cycle when the resonator is momentarily excited.} \]

This comes from two mechanisms:

1) Coupling Q: let \( R \to 0 \), and power lost to the feed line.

\[ \text{→ algebra → } Q_C = \frac{2L}{\omega_0^2 C^2} \]
2) "Internal Q": let $C_c \rightarrow 0$. Power lost to the resistor gives:

$$Q_i = \omega_0 R C$$

**What is $R$?**

- QP resistance
- Surface + dielectric losses
- Radiative losses
- E&M coupling to the box etc.

Some people break this into $Q_i + Q_{loss}$ or similar.

Together these determine the total $Q$:

$$Q_{R}^{-1} = Q_{c}^{-1} + Q_{i}^{-1} + \ldots$$

**What does this look like in practice?**
Above plot for a fixed $Q_c$

in IQ space: $Q$
Functional form is approx:

\[ S_{21} = 1 - \frac{Q_r}{Q_c} \left[ \frac{1}{1 + 2jQ_i X} \right] \]

\[ X = \frac{\omega - \omega_0}{\omega_0} \quad \text{and} \quad j = \sqrt{-1} \]

Complications:
- **Cable delay**
- **Non-matched impedances**
- **Amplitude, offsets, freq. dep. gain**

More general:

\[ Q_c \rightarrow \tilde{Q}_c = Q_c^p + jQ_c^i \]

\[ Q_c = |\tilde{Q}_c| \]

(Gao, etc. use \( Q_c e^{j\phi_0} \) notation)

To fit cable delay also, can add \( \alpha e^{-j\omega t} \) multiplier to whole thing

↑ normalization → cable delay
What happens when we shine light on this device?

$L_k$ and $R$ both change linearly in response to small amounts of optical power.

Resonator moves to the left and depth decreases. ($Q_i$ goes down.)

In most cases, the frequency signal is more sensitive.

We want to record \( \frac{\Delta f}{f_0} \) vs. time.
So- how do we actually take data?

1- do a sweep to find \( f_0, \phi(\Delta f) \)

\[
\text{NOTE THE POINTS GET FARTHER A-PART IN } |\Delta I^2 + \Delta Q^2|
\]

2- Place a tone at \( f_0 \), record \( I, Q(t) \) \( \rightarrow \phi(t) \).

We know \( \phi(f-f_0) \) from sweep.

Now \( f \) fixed, \( f_0 \) changes with power on the kid.

"Noise ball" \( \phi(t) \rightarrow \frac{\Delta f(t)}{f} \)
CONSIDERATIONS:

1. BIFURCATION.

Kinetic inductance is non-linear to 1st order.

\[ L_k(I) = L_k(0) \left[1 + \frac{I^2}{I_s^2}\right] \]

Since the current in the resonator is a function of \( f - f_0 \), this makes \( S_{21}(f) \) complicated and power dependent.

Eventually, \( S_{21}(f) \) has two solutions

\( I^* \) scales with: \( V_c, T_c, N_0 \)