Inflation in *Flatland*

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Flatland?

- Most of my time will be spent discussing the physics of inflation in 2+1 dimensions.
- Gravity is particularly simple in 2+1 dimensions; there are no local gravitational degrees of freedom — corresponds to no tensor modes in inflation (Deser, Jackiw, 't Hooft)
- The primary thing of interest is the relationship between cosmological adiabatic modes and asymptotic symmetries of de Sitter space
- Asymptotic symmetry algebra of 3d de Sitter space is well-known, and is very simple: two copies of the Virasoro algebra (Brown, Henneaux; Strominger)
- Another motivation: the scalar consistency relations in 3+1 dimensions can be traced back to the conformal symmetries of spatial slices, in 2+1 dimensions, this is infinite-dimensional

Soft theorems

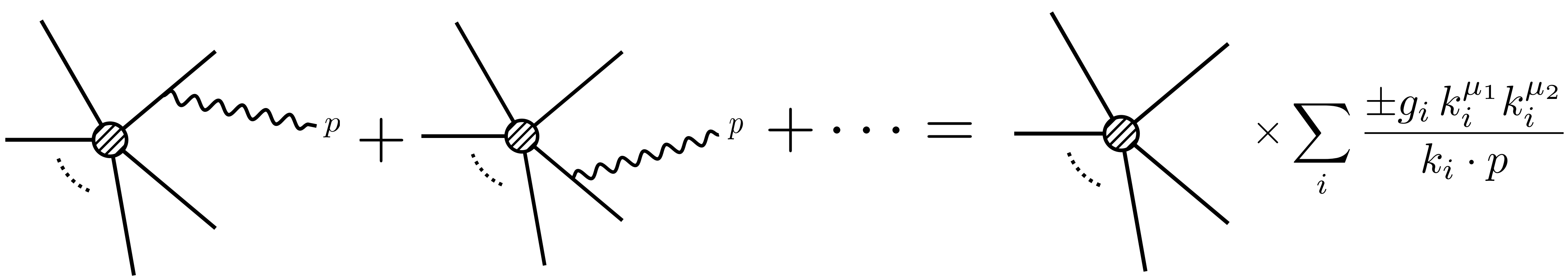
- The primary motivation for this type of investigation is to get a better understanding of soft theorems in cosmology

- Soft theorems are ubiquitous in theories with non-linearly realized symmetries

  e.g., Adler's zero in pion physics: $\lim_{q\to 0}\langle \pi^a_q \pi^{b_1}_{k_1}\cdots\pi^{b_n}_{k_n}\rangle = 0$

  Double-soft limit here is particularly interesting

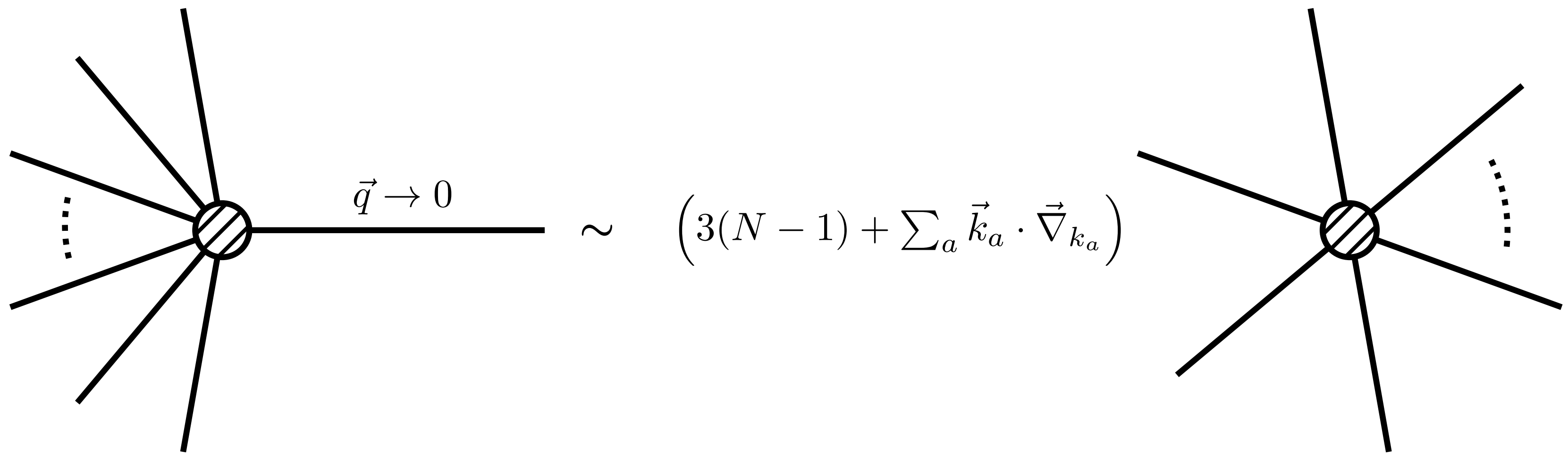
- Also play an important role in gauge theories (e.g., Weinberg's soft graviton theorem)



These identities have recently been understood as consequences of asymptotic symmetries in flat space (Strominger +many others)

Soft limits in cosmology (in 3+1d)

- Squeezed limits of cosmological correlators are related to symmetry-transformed lower-point functions



- e.g., Maldacena's consistency relation

$$\lim_{k_1 \to 0} \frac{1}{P_\zeta(k_1)} \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle' = -(n_s - 1) P_\zeta(k)$$

- These relations follow from symmetry, but to turn symmetry statements into correlator statements requires some assumptions

Soft limits in cosmology (in 3+1d)

- Hold in models of inflation satisfying technical assumptions:
  - there is only a single clock (roughly, single field)
  - modes start in the Bunch-Davies vacuum
  - ζ goes to a constant at long wavelengths

- These relations are sharp null-tests, and violations of them can be striking signatures of new physics (example: Quasi-single field inflation) (Chen, Wang; Akrani-Hamed, Maldacena)

$$\lim_{q\to 0} \frac{1}{P_\zeta(q)} \langle \zeta_q \zeta_k \zeta_k \rangle = -(n_s - 1) P_\zeta(k) + \mathcal{O}\left[\left(\frac{q}{k} \right)^\Delta P_s(\cos\theta) \right] \qquad \Delta = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

Standard derivation: “adiabatic modes”

- We are working in the context of a homogeneous FLRW background cosmology
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2$$
which we imagine is sourced by a scalar field for simplicity

- We now want to perturb both the metric and the scalar and study the perturbations

- In order to do this, we typically have to fix a gauge, a convenient choice is to use ADM variables
$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$$
and fix the spatial metric to be $h_{ij} = a^2(t)e^{2\zeta}\left(e^{\gamma}\right)_{ij}$
along with leaving the scalar unperturbed $\delta\phi = 0$

- Standard procedure: solve for lapse and shift, plug back in to the action to get an action for only ζ and γ

Standard derivation: “adiabatic modes”

- Even after gauge-fixing, there are always residual “large gauge transformations” which act as genuine symmetries

$$h_{ij} = a^2(t)e^{2\zeta}\left(e^{\gamma}\right)_{ij}$$

- Setting the tensors to zero for the time being, there are residual diffs:

$$\xi^i = \lambda(t)x^i \qquad\qquad \xi^i = 2b^j(t)x_jx^i - \vec{x}^2 b^i(t)$$

- These lead to a shift of ζ of the form:

$$\delta\zeta = \lambda(t)\left(1 + \vec{x}\cdot\vec{\nabla}\right)\zeta \qquad \delta\zeta = 2\vec{b}(t)\cdot\vec{x} + \left(2\vec{b}(t)\cdot\vec{x}x^i - \vec{x}^2 b^i(t)\right)\partial_i\zeta$$

these should be symmetries of the gauge-fixed action

Adiabatic modes

(Weinberg; Creminelli, Norena, Simonovic; Hinterbichler, Hui, Khoury)

- Actually, that is a bit too quick — there are constraint equations in GR and these diffs will not in general induce the correct lapse and shift

$$N^{(1)} = \frac{\dot{\zeta}}{H} \qquad\qquad N_i^{(1)} = -\frac{1}{H}\partial_i\zeta + \frac{a^2\epsilon}{c_s^2}\frac{\partial_i}{\nabla^2}\dot{\zeta}$$

- Demanding that these equations are preserved by a diff:

$$\delta\zeta = \frac{1}{3}\partial_i\xi^i \qquad\qquad \delta N^i = \dot{\xi}^i \qquad\qquad \delta N = 0$$

constrains the time dependence of the diff parameters:

$$\xi^i = \lambda x^i \qquad\qquad \xi^i = 2b^j x_j x^i - \vec{x}^2 b^i - 2b^i \int \frac{\mathrm{d}t}{aH}$$

- Interpretation: large diffs are inducing k = 0 profiles for ζ, the adiabatic mode condition ensures that these profiles are the zero momentum limit of a physical profile that solves the Einstein equations — similar story for tensors

From adiabatic modes to soft theorems

- Adiabatic modes are physical profiles that can be introduced by a diffeomorphism

- A local observer should not be able to tell whether or not they are in such a long-wavelength background

- A correlation function in the presence of a soft mode is related to a correlator without the soft mode, but in transformed coordinates

$$\langle\zeta(x_1)\cdots\zeta(x_n)\rangle_{\zeta_L} = \langle\zeta(\tilde{x}_1)\cdots\zeta(\tilde{x}_n)\rangle$$

- This leads to the inflationary soft theorems (can also be derived as Ward identities employing standard machinery)

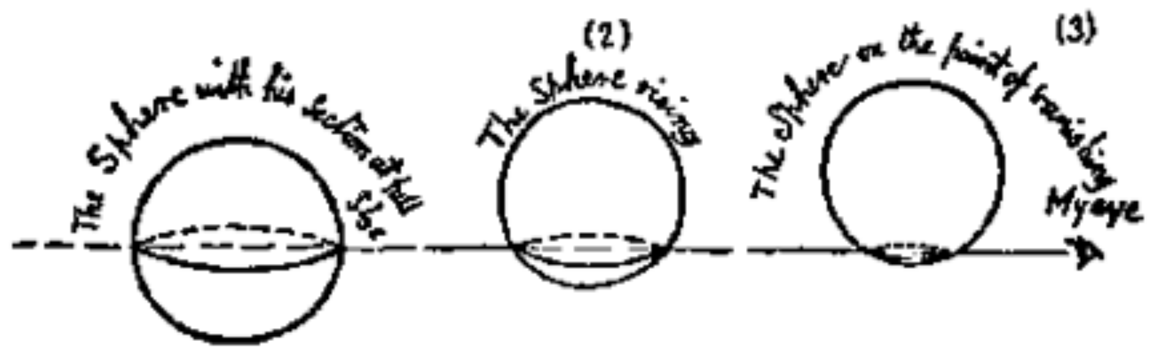
To three dimensions

Inflation in three dimensions

Some predictions:

- $r = 0$ (there are no local graviton perturbations) ✓

- You can't step over stuff… ✗



Nevertheless, it is an illuminating theoretical playground, so let's continue

Adiabatic modes in 2+1

- We would like to repeat the adiabatic mode construction in 3d, convenient to use complex coordinates
- The gauge choice $h_{ij} = e^{2\zeta}\left(e^{\gamma}\right)_{ij}$ for the spatial metric is preserved by any harmonic diff. $z \mapsto z + \xi^z$ $\quad \bar{z} \mapsto \bar{z} + \xi^{\bar{z}}$

$$\partial\bar{\partial}\xi^i = 0$$

- Harmonic functions can be split into holomorphic and anti-holomorphic parts

$$\xi^i = f^i(z,\eta) + \bar{g}^i(\bar{z},\eta)$$

- With general time dependence, these transformations will not satisfy the constraint equations, this restricts time dependence

$$\xi^z(z,\eta) = -z^{n+1}\,, \qquad \xi^{\bar{z}}(z,\eta) = -n(n+1)z^{n-1}\int_\eta^0 \frac{\mathrm{d}\eta'}{aH}$$

- There is also another type of allowed transformation: $\xi^z = \bar{g}^z(\bar{z})$ we'll come back to these.

Adiabatic puzzles

- This adiabatic mode condition is kind of puzzling — *any residual* diffeomorphism formally should be a symmetry of the gauge-fixed action

- Why are we forced to restrict these diffeomorphisms by imposing the adiabatic mode condition? Basically forces diffs. to induce physical modes

- It would be nice to think about this in a different way, hopefully to connect to something more fundamental

- There has been a lot of work on flat space soft theorems and their connections to asymptotic symmetries, is there a similar story here?

Asymptotic symmetries

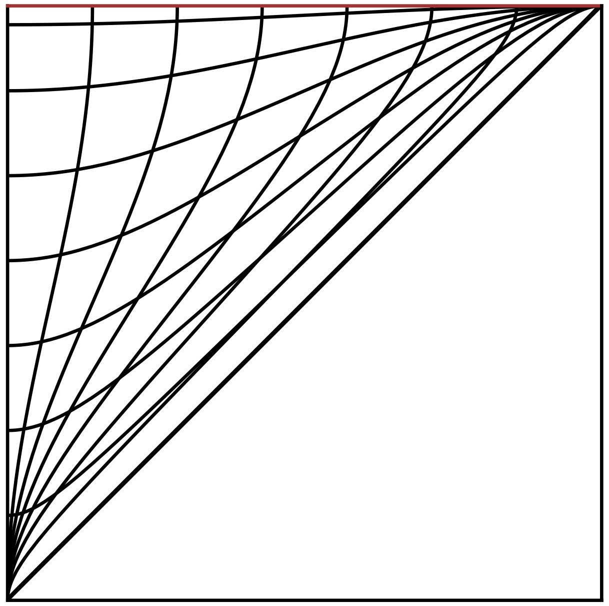
- Asymptotic symmetries: the group of diffeomorphisms which preserve the asymptotic structure of the theory (some boundary conditions)

  - *E.g.* in flat space for gravity, the asymptotic symmetry group is the set of all diffeomorphisms which preserve the structure at null infinity, form the BMS group

- The group that you get depends on the boundary conditions that you demand, there is no unique asymptotic symmetry group

- We will find that the adiabatic mode transformations that we just derived agree precisely with the asymptotic symmetries of dS_3

Asymptotic symmetry group of dS_3



- It is convenient to write de Sitter in terms of complex coordinates

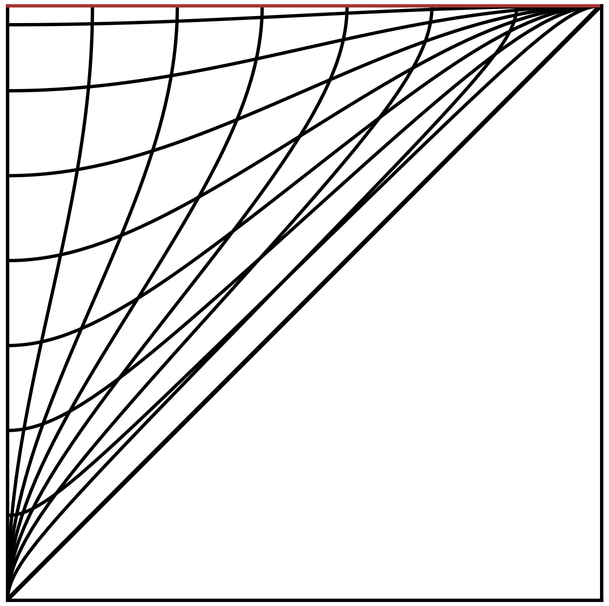
$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}z\mathrm{d}\bar{z}\right)$$

- To get the asymptotic symmetry group we impose some boundary conditions at temporal infinity $(\eta \to 0)$

- Ask for all diffeomorphisms which preserve the boundary conditions (Brown-Henneaux)

$$g_{\eta\eta} = -\frac{1}{H^2\eta^2} + \mathcal{O}(1)\,; \qquad g_{zz} = g_{\bar{z}\bar{z}} = \mathcal{O}(1)\,;$$
$$g_{z\bar{z}} = \frac{1}{2H^2\eta^2} + \mathcal{O}(1)\,; \qquad g_{\eta z} = g_{\eta\bar{z}} = \mathcal{O}(1)\,.$$

Asymptotic symmetry group of dS_3



- Most general diff. which does this is of the form (Strominger)

$$\xi = \frac{\eta}{2} U'(z)\partial_\eta + U(z)\partial + \frac{\eta^2}{2} U''(z)\bar{\partial}$$

plus an independent thing where everything is conjugated

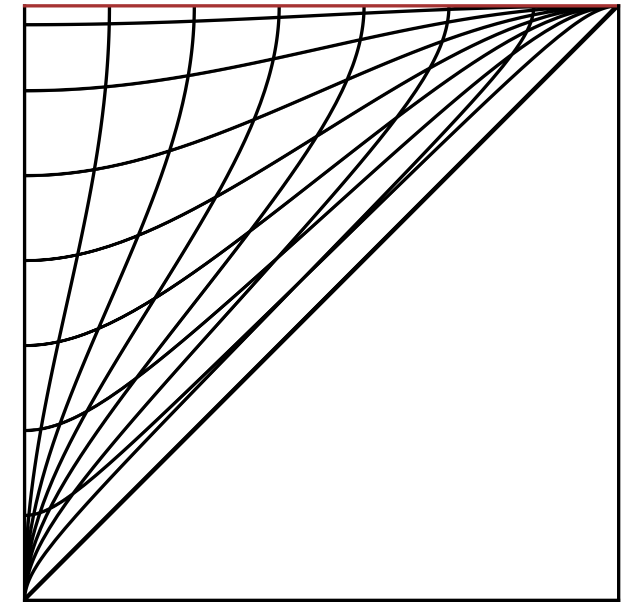
- Break this up into modes $U(z) = -\sum_n a_n z^{n+1}$

$$\ell_n = -\frac{(n+1)}{2} z^n \eta \partial_\eta - z^{n+1}\partial - \frac{\eta^2}{2} n(n+1) z^{n-1}\bar{\partial}$$

these satisfy the algebra $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$

there is also an anti-holomorphic copy, two copies of Witt algebra, which gets centrally extended to 2 Virasoros

Asymptotic symmetry group of dS_3



- Recall adiabatic modes

$$\xi^z(z,\eta) = -z^{n+1}\,, \qquad \xi^{\bar{z}}(z,\eta) = -n(n+1)z^{n-1}\int_\eta^0 \frac{\mathrm{d}\eta'}{aH}$$

- Asymptotic symmetries

$$\xi^\eta = -\frac{(n+1)}{2}z^n\eta \qquad \xi^z = -z^{n+1} \qquad \xi^{\bar{z}} = -\frac{\eta^2}{2}n(n+1)z^{n-1}$$

- These are the same vector fields as the ones which generate adiabatic modes!

- However, do not reproduce the "tensor" adiabatic modes — these require different boundary conditions

Interpretation

- After doing one of these diffeomorphisms, the line element looks like

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}z\mathrm{d}\bar{z}\right) + \frac{1}{2H^2}\partial^3 U(z)\mathrm{d}z^2$$

- This is a physically distinct configuration, differing by insertions of soft gravitons

- Like spontaneous symmetry breaking, we are mapped to an equivalent vacuum

- This connects with the adiabatic mode condition, asymptotic symmetries are naturally the transformations that introduce physical long-wavelength modes

Action on the curvature perturbation

How do these symmetries act on the inflationary curvature perturbation?

- We know the adiabatic modes, so we could work this out using standard techniques, but it is interesting to see directly from the asymptotic symmetry perspective.

Consider the perturbed metric

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + e^{2\zeta(\eta,z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}\right)$$

- If zeta is a function of just z, this metric is asymptotically dS (after coordinate transformations)

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}z\mathrm{d}\bar{z}\right) + \frac{1}{H^2}\partial^2\zeta\mathrm{d}z^2$$

Action on the curvature perturbation

- This is precisely of the form that can be induced by an asymptotic symmetry transformation

- A symmetry ℓ_n inserts a profile of the form $\zeta \sim z^n$

$$\delta_n \zeta = -\frac{1}{2}(n+1)z^n - z^{n+1}\partial\zeta$$

- In terms of Cartesian coordinates, these symmetries look like

$$\delta_{i_1 \cdots i_n}\zeta \propto x^{(i_1} \cdots x^{i_n)_{\rm T}} + \mathcal{O}(\zeta)$$

- These transformations for ζ are symmetries of the gauge-fixed action, should have corresponding soft theorems

- Note that there is an infinite number of scalar symmetries, in contrast to higher dimensions

Soft-ζ Theorems

The nonlinearly realized symmetries lead to Ward identities relating (N+1) and N-point functions.

$$\langle\Omega|[Q_n,\mathcal{O}]|\Omega\rangle = -i\langle\Omega|\delta_n\mathcal{O}|\Omega\rangle$$

- Explicitly this takes the form

$$\lim_{q,\bar{q}\to 0}\frac{\partial^n}{\partial\bar{q}^n}\left(\frac{1}{2P_\zeta(q,\bar{q})}\langle\zeta_{q,\bar{q}}\,\mathcal{O}\rangle'_c\right) = -\sum_{a=1}^{N-1}\left(\frac{\partial^n}{\partial\bar{k}_a^n}+\frac{1}{(n+1)}\bar{k}_a\frac{\partial^{n+1}}{\partial\bar{k}_a^{n+1}}\right)\langle\mathcal{O}\rangle'_c$$

- Simplest one that has no direct higher-dimensional analogue (n=2 relating 3 and 2 point function)

$$\lim_{q,\bar{q}\to 0}\frac{\partial^2}{\partial\bar{q}^2}\left(\frac{1}{2P_\zeta(q,\bar{q})}\langle\zeta_{q,\bar{q}}\zeta_{k_2,\bar{k}_2}\zeta_{k_3,\bar{k}_3}\rangle\right) = -\left(\frac{\partial^2}{\partial\bar{k}^2}+\frac{1}{3}\bar{k}\frac{\partial^3}{\partial\bar{k}^3}\right)\langle\zeta_{k,\bar{k}}\zeta_{k,\bar{k}}\rangle'$$

A check

- In order to explicitly check this relation, it is convenient to use EFT of inflation formalism (Creminelli et al.; Cheung et al.)

$$S = \int \mathrm{d}^d x \mathrm{d}t \sqrt{-g} \left(\frac{M_{\rm Pl}^{(d-1)}}{2} R - c(t) g^{00} - \Lambda(t) + \frac{M_2(t)}{2} (\delta g^{00})^2 + \frac{M_3(t)}{3!} (\delta g^{00})^3 + \dots \right) ,$$

- work in the decoupling limit

$$M_{\rm Pl} \to \infty \qquad \dot{H} \to 0 \qquad M_{\rm Pl}^{(d-1)} \dot{H} = \text{fixed}$$

$$S = -\frac{(d-1) M_{\rm Pl}^{(d-1)} \dot{H}}{2 c_s^2} \int \mathrm{d}^d x \mathrm{d}t \, a^d \left(\dot{\pi}^2 - \frac{c_s^2}{a^2} (\nabla \pi)^2 + (c_s^2 - 1) \frac{1}{a^2} \dot{\pi} (\nabla \pi)^2 \right) .$$

- squeezed limit

$$\lim_{k_1 \to 0} \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle' = \frac{(D-1)}{3} P_\zeta(k_1) P_\zeta(k_2) (1 - c_s^{-2}) \left((D-2) \frac{k_1^2}{k_2^2} - \frac{(D-3)(D+1)}{4} \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_2^4} \right)$$

Conclusions

- Asymptotic symmetries and scalar adiabatic modes seem to be in one-to-one correspondence (at least in this simplified setting)
- It would be nice to make this same kind of explicit connection in 3+1 dimensions
- "Tensor" adiabatic modes seem to require less stringent boundary conditions, would be nice to understand what these are. More generally, it is not clear how adiabatic modes pick out preferred BCs
- 3d inflation provides a nice simplified setting to explore many issues; correlation functions satisfy an infinite number of soft theorems involving only the scalar perturbation
- There should be a dual CFT derivation/interpretation of these results, along with extension to multi-field scenarios