An Estimator for statistical anisotropy from the CMB bispectrum

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1. CMB physics
2. Non-Gaussianity
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5. Anisotropic non-Gaussianity
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...based on:

N. Bartolo, E. D., M. Liguori, S. Matarrese, A. Riotto
JCAP 1201:029

N. Bartolo, E. D., S. Matarrese, A. Riotto
JCAP 0910:015, *JCAP 0911:028*, *Adv.Astron.2010:752670*

Cosmic Microwave Background (CMB)

The **CMB** is a window on the Early Universe physics.

The statistical properties of its fluctuations (e.g. in temperature) are well-explained within the theory of **inflation**: the quantum fluctuations of fields during inflation are the seeds of the CMB anisotropies we observe today!

Temperature fluctuations

$\Delta T(\hat{n}) \equiv T(\hat{n}) - T_0$ random field

on a 2D sphere ($\hat{n} = (\phi, \theta)$ identifies a point of the sphere)

$T_0 = \int \frac{d\Omega_{\hat{n}}}{4\pi} T(\hat{n})$ is the average temperature

CMB Temperature Anisotropies

Primordial fluctuations

metric: $g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$
inflaton field: $\phi(\vec{x},t) = \phi^{(0)}(t) + \delta\phi(\vec{x},t)$
ζ gauge invariant combination of scalar metric and energy-momentum tensor fluctuations
$\zeta \simeq \frac{\Delta T}{T_0}$

Temperature fluctuations in harmonic space

$\frac{\Delta T}{T_0}(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$
$a_{lm} \simeq 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Delta_l(k)\zeta(\vec{k})Y_{lm}(\hat{k})$
$\Delta_l(k)$ radiation transfer function.

CMB Temperature Anisotropies

We study the temperature fluctuations by looking at its n-point correlations:

$$\langle \frac{\Delta T(\hat{n}_1)}{T_0} \frac{\Delta T(\hat{n}_2)}{T_0} \frac{\Delta T(\hat{n}_3)}{T_0} \ldots \rangle$$

$\langle \ldots \rangle$ denotes the ensemble average, i.e. the average over all possible field configurations.

CMB temperature anisotropies

From the CMB, we find indications that primordial fluctuations are, among other things,...

- compatible with being Gaussianly distributed
- consistent with being statistically isotropic

What do we mean by "Gaussian"...

Temperature Anisotropies are Gaussian if their Probability Density Function **(PDF)** is given by

$$P(\Delta T) = \frac{1}{(2\pi)^{N_{pix}/2}|\xi|^{1/2}} exp\left[-\frac{1}{2}\sum_{ij}\Delta T_i\left(\xi^{-1}\right)_{ij}\Delta T_j\right]$$

$\Delta T_i \equiv \Delta T(\hat{n})$, N_{pix} number of pixels on the sky, $|\xi| = det[\xi]$

$\xi_{ij} \equiv \langle \Delta T_i \Delta T_j \rangle$ covariance matrix (or 2-point correlation function)

In **Harmonic Space**

$$P(a) = \frac{1}{(2\pi)^{N_{harm}/2}|C|^{1/2}} exp\left[-\frac{1}{2}\sum_{lm,l'm'} a_{lm}\left(C^{-1}\right)_{lm,l'm'} a_{l'm'}\right]$$

N_{harm} is the number of l and m, $C_{lm,l'm'} = \langle a_{lm} a^*_{l'm'} \rangle$

A Gaussian random field is completely characterized by its 2-point correlation function

What do we mean by "Gaussian"...

Similarly, for Gaussian primordial field, the only relevant statistical quantity is the **2-point function** or, in momentum space, the **power spectrum**

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle$$

$\zeta(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \zeta_{\vec{k}}(t)$

...and by "non-Gaussian"?

Any deviation from a Gaussian PDF is called "non-Gaussianity". In the weakly non-Gaussian case, the PDF can be written as an expansion around the Gaussian PDF (**Edgeworth expansion**)

$$\begin{aligned} P(a) &= \Bigg(1 - \frac{1}{3!}\sum_{l_i,m_i} C^{l_1 l_2 l_3}_{m_1 m_2 m_3} \frac{\partial}{\partial a_{l_1 m_1}} \frac{\partial}{\partial a_{l_2 m_2}} \frac{\partial}{\partial a_{l_3 m_3}} \\ &+ \frac{1}{4!}\sum_{l_i,m_i} C^{l_1 l_2 l_3 l_4}_{m_1 m_2 m_3 m_4} \frac{\partial}{\partial a_{l_1 m_1}} \frac{\partial}{\partial a_{l_2 m_2}} \frac{\partial}{\partial a_{l_3 m_3}} \frac{\partial}{\partial a_{l_4 m_4}} \\ &- \frac{1}{5!}\sum_{l_i,m_i} C^{l_1 l_2 l_3 l_4 l_5}_{m_1 m_2 m_3 m_4 m_5} \frac{\partial}{\partial a_{l_1 m_1}} \frac{\partial}{\partial a_{l_2 m_2}} \frac{\partial}{\partial a_{l_3 m_3}} \frac{\partial}{\partial a_{l_4 m_4}} \frac{\partial}{\partial a_{l_5 m_5}} \\ &+ \ldots\Bigg) \times \frac{e^{-\frac{1}{2}\sum_{l_q,m_q} a^*_{l_4 m_4} C^{-1}_{l_4 m_4, l_5 m_5} a_{l_5 m_5}}}{(2\pi)^{N_p/2} (detC)^{1/2}}, \end{aligned}$$

$C^{l_1 l_2 l_3 \ldots}_{m_1 m_2 m_3 \ldots} \equiv \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \ldots \rangle$ higher moments of the distribution

...and by "non-Gaussian"?

Similarly, for non-Gaussian primordial fields we expect non-zero (connected) correlation functions of order higher than two e.g. :

- **Bispectrum** $\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle$
- **Trispectrum** $\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\zeta(\vec{k}_4)\rangle$

Why is non-Gaussianity so interesting?

- Primordial fluctuations are very close to being Gaussian but we don't know if they are exactly Gaussian, so **non-Gaussianity needs to be furtherly constrained**
- Non-Gaussianity predictions can be different for different models of inflation which are consistent with the current results for the power spectrum (e.g. scale invariance). Measuring **non-Gaussianity would be a huge help in identifying the correct model for the Early Universe**.

Statistical Isotropy and Homogeneity

They have both been employed as working assumptions in obtaining the most important results in modern cosmology

What it means: statistical expectation values are assumed to be invariant under spatial rotations and translations, for instance:

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2)P_\zeta(k_1)$$

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_\zeta(k_1,k_2,k_3)$$

Notice: power spectrum is rotationally invariant, bispectrum is preserved under rotations of the triangle of vertices $\vec{k}_1, \vec{k}_2, \vec{k}_3$

Statistical Isotropy

Similary, for the temperature fluctuation field, the angular correlation function is **invariant w.r.t. spatial rotations**:

$$\langle \frac{\Delta T(R\hat{n}_1)}{T_0} \frac{\Delta T(R\hat{n}_2)}{T_0} \frac{\Delta T(R\hat{n}_3)}{T_0} \dots \rangle = \langle \frac{\Delta T(\hat{n}_1)}{T_0} \frac{\Delta T(\hat{n}_2)}{T_0} \frac{\Delta T(\hat{n}_3)}{T_0} \dots \rangle$$

$R = R(\alpha, \beta, \gamma)$ rotation matrix for the Euler angles α, β, γ.

As a result...

$$\langle a_{l_1 m_1} a^*_{l_2 m_2} \rangle = (-1)^{m_1} C_{l_1} \delta_{l_1 l_2} \delta_{m_1 m_2}$$

C_l "angular power spectrum"

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = B_{l_1 l_2 l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$B_{l_1 l_2 l_3}$ "angular bispectrum"

Statistical Anisotropy

Let us relax the assumption of statistical isotropy; then...

$$P_\zeta(k) \longrightarrow P_\zeta(\vec{k})$$

$$B_\zeta(k_1, k_2, k_3) \longrightarrow B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3)$$

$$C_{lm,l'm'} \equiv \langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'} + \Delta_{lml'm'} \rightarrow \text{non diagonal variance!}$$

$$B^{l_1 l_2 l_3}_{m_1 m_2 m_3} \equiv \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_4} \rangle \neq B_{l_1 l_2 l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Is Statistical Isotropy the right assumption?

- it is **reasonable**, given how our Universe looks on large scales
- it is **convenient** for data analysis (e.g. it allows to average out temperature fluctuations from different regions of the sky, leading to accurate constraints on the cosmological parameters)

Nevertheless...

...there is no a priori reason why this assumption shoud be the correct one, so it is not right to take it for granted before it is accurately tested!

More motivation for testing statistical isotropy: CMB Anomalies

- asymmetry in the large scale power spectrum and in higher order correlators between northern and southern ecliptic hemisphere;
  (Eriksen et al, 04')
- alignment of quadrupole ($l = 2$) and octupole ($l = 3$) modes of temperature anisotropies;
  (Bennett et al., 96')
- cold spots (i.e. regions of suppressed power);
  (Vielva et al., 04')
- lack of correlation at large angles;
  (Spergel et al., 07')

Are these observations telling us that statistical isotropy is violated? If yes, are these effects cosmological?

We know what is most likely NOT cosmological, e.g. the quadrupolar anisotropy

A widely used parametrization, valid for most parity conserving vector field models of inflation which involve only one preferred spatial direction

$$P_\zeta(\vec{k}) = P_\zeta(k)\left[1 + G(k)\left(\hat{k}\cdot\hat{n}\right)^2\right]$$

(Ackermann-Carroll-Wise, 07')

Data Analysis

From CMB (WMAP5 data):

$G = 0.29 \pm 0.031$
(Groeneboom et al, 2009; Hanson and Lewis, 2009)

Power anisotropy is found to be compatible with beam asymmetries, uncorrected in the maps (Hanson-Lewis, 2010)

Statistical Anisotropy

More observations and more data analysis are needed in order to understand CMB anomalies and in order to understand whether or not statistical isotropy is the right working assumption.

On the theoretical side, are there any consistent models for the Early Universe that are able to predict statistical anisotropy?

Statistical Anisotropy from Cosmology

some of the existing models...

- pre-inflationary and inflationary Bianchi type cosmologies;
- anisotropic dark energy models;
- models of inflation with vector fields
  (vector inflation, vector curvaton, models with varying kinetic functions)

  Hybrid inflation model with vector field
  (Yokoyama and Soda, 2011)

  $\mathcal{L} \simeq \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + \partial_\mu\chi\partial^\mu\chi\right) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu} + V(\phi, \chi, B_\mu)$
  ($\phi \equiv$ inflaton, $\chi \equiv$ waterfall field)

  This model can easily predict a negligible anisotropy for the power spectrum vs a large anisotropy for the bispectrum

Bispectrum analysis

Various vector field models of inflation predict observable levels of non-Gaussianity:

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = B_\zeta^{(isotropic)}(k_1, k_2, k_3) + B^{(anisotropic)}(\vec{k}_1, \vec{k}_2, \vec{k}_3)$$

$$\rightarrow \frac{6}{5} f_{NL} \equiv \frac{B_\zeta(\vec{k}, \vec{k}_2, \vec{k}_3)}{P_\zeta(\vec{k}_1) P_\zeta(\vec{k}_2) + 2perms.} = \frac{6}{5}\left(f_{NL}^{(isotropic)} + f_{NL}^{(anisotropic)}\right)$$

Anisotropy parameters

$$\lambda \simeq \frac{f_{NL}^{(anisotropic)}}{f_{NL}^{(isotropic)}}$$

- complementary to probes of s. a. features from the power spectrum
- important for models which predict a negligible level of statistical anisotropy in the power spectrum but observable in the bispectrum

  $\rightarrow$ Searches for signatures of statistical anisotropy limited to the power spectrum could be blind to such bispectrum features

Step 1: Primordial Bispectrum

Build a template ζ bispectrum corresponding to the desired vector field model(s):

$$\begin{aligned} B(\vec{k}_1,\vec{k}_2,\vec{k}_3) &= B^{iso}(k_1,k_2)\Big[1+g_B(k_1,k_2)\Big(p(k_1)\left(\hat{k}_1\cdot\hat{N}_A\right)^2+p(k_2)\left(\hat{k}_2\cdot\hat{N}_A\right)^2 \\ &+ p(k_1)p(k_2)\left(\hat{k}_1\cdot\hat{k}_2\right)\left(\hat{k}_2\cdot\hat{N}_A\right)\left(\hat{k}_1\cdot\hat{N}_A\right)\Big)\Big]+2\ perms. \end{aligned}$$

$$B^{iso}(k_1,k_2)=\frac{6}{5}f_{\rm NL}\frac{A^2}{k_1^3k_2^3}$$

valid for most single vector field models (e.g. Soda and Yokoyama)

Step 2: CMB Bispectrum

Compute the CMB bispectrum from the primordial one and identify the parameters that encode the information about statistical anisotropy

$$
\begin{aligned}
B^{l_1\ l_2\ l_3}_{m_1 m_2 m_3} &\equiv \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = B^{l_1 l_2 l_3 (I)}_{m_1 m_2 m_3} + B^{l_1 l_2 l_3 (A)}_{m_1 m_2 m_3} \\
&= f_{NL} \left(B^{l_1 l_2 l_3 (I)}_{m_1 m_2 m_3} |_{f_{NL}=1} + \sum_{LM} \lambda_{LM} B^{l_1 l_2 l_3 (A) LM}_{m_1 m_2 m_3} |_{f_{NL}=1} \right)
\end{aligned}
$$

$$
\begin{aligned}
B^{l_1 l_2 l_3 (I)}_{m_1 m_2 m_3} &= b_{l_1 l_2 l_3} G^{l_1 l_2 l_3}_{m_1 m_2 m_3} \\
B^{l_1 l_2 l_3 (A)}_{m_1 m_2 m_3} &= \left(\frac{18}{5\pi} \right)^3 \sum_{l_1' m_1'} \sum_{LM} \lambda_{LM} G^{l_1' l_2 l_3}_{m_1' m_2 m_3} G^{l_1 l_1' L}_{m_1 - m_1' M} (-1)^{l_1} (i)^{l_1 + l_1'} (-1)^{m_1'} b^{l_1'}_{l_1 l_2 l_3} \\
b^{l_1'}_{l_1 l_2 l_3} &\equiv \int dx x^2 \int dk_1 dk_2 dk_3 (k_1 k_2 k_3)^2 \Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) \\
&\times \; j_{l_1'}(k_1 x) j_{l_2}(k_2 x) j_{l_3}(k_3 x) \left(B^{iso}(k_1, k_2) + B^{iso}(k_1, k_3) \right)
\end{aligned}
$$

Step 3: Optimal and Unbiased Estimators for λ_{LM}

$$\hat{\lambda}_{LM} = \frac{1}{F_{\lambda_{LM}\lambda_{LM}}}\left(f_{NL}B^{(A)LM} - f_{NL}^2 B^{(I)} B^{(A)LM}\right)$$

$$B^{(I)} \equiv \frac{1}{6}\sum_{l_i m_i} B^{l_1 l_2 l_3 (I)}_{m_1 m_2 m_3}|_{f_{NL}=1}\left(\frac{a^*_{l_1 m_1} a^*_{l_2 m_2} a^*_{l_3 m_3}}{C_{l_1} C_{l_2} C_{l_3}} - \frac{(-1)^{m_2}}{C_{l_1} C_{l_2}}\delta^{l_3}_{l_2}\delta^{-m_3}_{m_2} a^*_{l_1 m_1} + \dots\right)$$

$$B^{(A)LM} \equiv \frac{1}{6}\sum_{l_i m_i} B^{l_1 l_2 l_3 (A)LM}_{m_1 m_2 m_3}|_{f_{NL}=1}\left(\frac{a^*_{l_1 m_1} a^*_{l_2 m_2} a^*_{l_3 m_3}}{C_{l_1} C_{l_2} C_{l_3}} - \frac{(-1)^{m_2}}{C_{l_1} C_{l_2}}\delta^{l_3}_{l_2}\delta^{-m_3}_{m_2} a^*_{l_1 m_1} + \dots\right)$$

Fisher matrix is diagonal

$$F_{\lambda_{LM}\lambda_{L'M'}} \equiv -\left\langle \frac{\partial^2 \ln(P)}{\partial\lambda_{LM}\partial\lambda^*_{L'M'}}\right\rangle = f_{NL}^2\left\langle B^{A(LM)} B^{A(LM)*}\right\rangle \delta_{LL'}\delta_{MM'}$$

Step 3: Fisher Errors

$\sigma^2_{\lambda_{LM}} = (F^{-1})_{\lambda_{LM}\lambda_{LM}}$

For a quadrupolar anisotropy (L=2):

- $\frac{1}{\sigma_{\lambda_{2M}}} \simeq 0.4 f_{NL} \left(\frac{l}{2000}\right)$

  valid for an experiment up to $l \simeq 2000$

- $\sigma_{\lambda_{2M}} \simeq 0.1$

  for $f_{NL} \simeq 32$ (around the current central value for the local limit) and for $l \simeq 1500$

Conclusions

- observation of "anomalies" in the CMB has forced to reconsider the assumption of statistical isotropy for cosmological correlation functions and to further test it
- theories of inflation with vector fields and with statistical anisotropy predictions have attracted a lot of attention recently, also motivated by the possibility that some of these "anomalies" might be a hint of rotational invariance breaking at very early times
- the quadrupolar power spectrum anisotropy is most likely not cosmological in origin; no such analysis has been yet performed for the bispectrum, so one cannot exclude that statistical anisotropy features may be present in the bispectrum that are hidden from the power spectrum; this particularly motivates models that predict an isotropic two-point statistics vs an anisotropic bispectrum

Conclusions

- we present an estimator for some multipole coefficients λ_{LM}, which represent the ratio of the anisotropic to isotropic bispectrum amplitudes
- we compute the Fisher matrix for the λ_{LM} coefficients and we show that with an experiment like Planck, anisotropy in the bispectrum would be detectable for a relative amplitude of the anisotropic to the isotropic part $\geq 10\%$ if the non-linearity parameter $f_{NL} \simeq 30$