Black hole entropy of gauge fields

William Donnelly (University of Waterloo)
with Aron Wall (UC Santa Barbara)

September 29th, 2012
Black hole entropy

Black holes possess an entropy given by the Bekenstein-Hawking formula:

\[ S_{BH} = \frac{Ac^3}{4G\hbar}. \]

Can be inferred \textit{macroscopically}.

1. From the Hawking temperature \( T_H = \frac{\hbar}{2\pi\kappa} \) and the first law of black hole mechanics \( \delta E = T_H\delta S_{BH}. \)

2. From a saddle point evaluation of the Euclidean partition function.\(^2\)

But the statistical meaning of \( S_{BH} \) is not clear.

What is the statistical mechanics of black hole thermodynamics?

\(^1\)Hawking 1975
\(^2\)Gibbons & Hawking 1977
Entanglement entropy

The black hole horizon also has an entanglement entropy.\(^3\)

We have a tensor product decomposition, and partial trace

\[ \mathcal{H} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}, \quad \rho_{\text{out}} = \text{tr}_{\text{in}} |\psi\rangle\langle\psi| . \]

The entropy of \( \rho_{\Omega} \) is the entanglement entropy:

\[ S_{\text{ent}} = - \text{tr} \rho_{\text{out}} \ln \rho_{\text{out}} . \]

Statistical meaning: entropy comes from missing correlations due to inaccessible black hole interior.

How is this related to the macroscopic quantity \( S_{\text{BH}} \)?

\(^3\)Sorkin 1983; Bombelli, Koul, Lee, & Sorkin 1986; Srednicki 1993
How are $S_{\text{BH}}$ and $S_{\text{ent}}$ related?

To relate $S_{\text{BH}}$ and $S_{\text{ent}}$, we use the conical method.\(^4\)

For simplicity we consider a Rindler horizon.

Let $Z(\beta)$ be the Euclidean path integral on a cone of angle $\beta$, times $D-2$ flat and compact transverse directions.

\[
Z = \int D\phi \, e^{-S[\phi]}
\]

\[
S_{\text{cone}} \equiv \left( 1 - \beta \frac{\partial}{\partial \beta} \right) \ln Z \bigg|_{\beta=2\pi}
\]

This conical entropy is equivalent to varying the period at $\infty$.

**Strategy:** Compute $S_{\text{cone}}$ in effective theory, and in the quantum theory.

\(^4\)Bañados, Teitelboim & Zanelli 1993
Suppose we integrate out the matter fields, leading to an effective action

\[ \int D\phi e^{-S[\phi]} = e^{-\int \sqrt{g} L_{\text{eff}}}, \quad L_{\text{eff}} = \frac{1}{16\pi G_{\text{eff}}} (R - 2\Lambda_{\text{eff}} + \ldots) \]

The conical entropy formula gives the Bekenstein-Hawking formula \(^5\)

\[ S_{\text{cone}} = \frac{A}{4G_{\text{eff}}} = S_{\text{BH}}. \]

In terms of the effective Newton’s constant, \(G_{\text{eff}}\).

---

\(^5\)Susskind & Uglum 1994; Jacobson 1994
Now consider evaluating the entropy in the quantum theory.

The Minkowski vacuum $|0\rangle$ restricted to one Rindler wedge is thermal in the boost generator $K$:

$$\rho_R \equiv \text{tr}_L |0\rangle \langle 0| = \frac{e^{-2\pi K}}{Z(2\pi)},$$

$$Z(\beta) \equiv \text{tr} e^{-\beta K}.$$

Varying $\beta$ is equivalent to varying the temperature of a thermal state:

$$S_{\text{cone}} \equiv \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z \bigg|_{\beta=2\pi} = -\text{tr} \rho_R \ln \rho_R = S_{\text{ent}}.$$

---

6 Bisognano & Wichmann 1975
Not so fast!

The conical geometry also has a singular curvature at the tip:\(^7\)

\[ R_{abcd}(x) = (2\pi - \beta)\epsilon_{ab}\epsilon_{cd}\delta\Sigma(x). \]

Nonminimally coupled matter interacts with this curvature.

The contribution to the conical entropy coming from the tip is:

\[ \langle S_{\text{Wald}} \rangle = -2\pi \int_\Sigma \sqrt{h} \langle \frac{\partial L}{\partial R_{abcd}} \rangle \epsilon_{ab}\epsilon_{cd}. \]

This term is the contribution of the matter fields to the Wald entropy.\(^8\)

Thus for nonminimally coupled matter,\(^9\)

\[ S_{\text{BH}} = S_{\text{ent}} + \langle S_{\text{Wald}} \rangle. \]

---

\(^7\)Fursaev & Solodukhin 1995

\(^8\)Wald 1993; Visser 1993; Jacobson, Kang & Myers 1993

\(^9\)As suggested by arguments in Frolov & Fursaev 1997
The conical entropy has been calculated for free fields of spin \( \leq 2 \),

\[
S_{\text{cone}} = A \ c_1 \left( 2\pi \int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{(4\pi s)^{D/2}} \right).
\]

\( c_1 \) depends on the field and \( N \), the number of on-shell degrees of freedom\(^{10}\)

<table>
<thead>
<tr>
<th>Spin</th>
<th>Field</th>
<th>( N )</th>
<th>( c_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Nonminimally coupled scalar</td>
<td>1</td>
<td>( \frac{N}{6} ) – ( \xi )</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>Dirac spinor</td>
<td>( 2 \lfloor \frac{D}{2} \rfloor - 1 )</td>
<td>( \frac{N}{6} )</td>
</tr>
<tr>
<td>1</td>
<td>Maxwell field</td>
<td>( D - 2 )</td>
<td>( \frac{N}{6} ) – 1</td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>Rarita-Schwinger field</td>
<td>( (D - 3)2 \lfloor \frac{D}{2} \rfloor - 1 )</td>
<td>( \frac{N}{6} )</td>
</tr>
<tr>
<td>2</td>
<td>Graviton</td>
<td>( \frac{D(D-3)}{2} )</td>
<td>( \frac{N}{6} ) – ( \frac{D^2-D+4}{2} )</td>
</tr>
</tbody>
</table>

For gauge fields there is a mysterious **contact term**.\(^{11}\)

---

\(^{10}\)Solodukhin 2011

\(^{11}\)Kabat 1995
For the electromagnetic field the Lagrangian is

$$L = \frac{1}{4} F^{ab} F_{ab}, \quad \Rightarrow \quad \text{expect} \quad S_{\text{Wald}} = 0.$$ 

We add ghosts $c$ and $\bar{c}$, a gauge fixing term, and integrate by parts:

$$L' = -\frac{1}{2} A^a (g_{ab} \nabla^2 - R_{ab}) A^b - \bar{c} \nabla^2 c.$$ 

The Wald entropy contribution from the gauge field is:

$$\langle S_{\text{Wald}} \rangle = -\pi \int_{\Sigma} \sqrt{h} \, g^{ab} \langle A_a A_b \rangle.$$ 

Evaluated using the heat-kernel regularization it gives $c_1 = -1$.

**Problem:** Gauge invariance? What about $D = 2$, where there are no local degrees of freedom?
Compact spacetime

We now consider $D = 2$ and compactify (e.g. 2D de Sitter).

In 2D, any vector field can be written as

$$A = d\phi + \delta \psi + B, \quad \Delta B = 0$$

The vector field cancels with the ghosts up to zero modes.

The number of zero modes (vector minus two ghosts) is $2g - 2 = -\chi$, where $\chi$ is the Euler characteristic.

Using Gauss-Bonnet, we can write $\chi = \frac{1}{4\pi} \int \sqrt{g} R$.

Zero mode contribution to the effective action is proportional to $\int \sqrt{g} R$

$$S_{\text{zero modes}} = - \left( 2\pi \int_{e^2}^{\infty} ds \frac{e^{-m^2 s}}{(4\pi s)^{D/2}} \right).$$

The $c_1 = -1$ in the conical entropy comes from zero modes.
Reduced phase space

Heat kernel method does not treat zero modes properly.

2D gauge theory has a huge symmetry group: area-preserving diffeomorphisms. It is “almost topological” and can be solved exactly.\footnote{Witten 1991}

\[
Z = \sum_{E \in q\mathbb{Z}} e^{-\frac{1}{2}VE^2}.
\]

Since \( V \propto \beta \), the conical entropy is

\[
S_{\text{cone}} = \left(1 - V \frac{\partial}{\partial V}\right) \ln Z = -\sum_E p_E \ln p_E = S_{\text{ent}}.
\]

This is finite, positive, and equal to the entanglement entropy.

One can show that \( S_{\text{cone}} = S_{\text{ent}} \) for 2D Yang-Mills as well.

\footnote{Witten 1991}
Conclusions:

- Black hole entropy is closely related to entanglement entropy

\[ S_{BH} = S_{\text{ent}} + \langle S_{\text{Wald}} \rangle. \]

- But the “contact term” in the entropy of gauge fields can’t be explained this way; it is absent when the partition function is evaluated carefully (in \( D = 2 \)).

Future work:

- Gauge theories in \( D > 2 \).
- Linearized gravity.

For details see arXiv:1206.5831.

Thank you.