Black hole entropy of gauge fields

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Black hole entropy

Black holes possess an entropy given by the Bekenstein-Hawking formula:

$$S_{\mathrm{BH}} = \frac{Ac^3}{4G\hbar}.$$

Can be inferred **macroscopically**.

- From the Hawking temperature $T_{\mathrm{H}} = \frac{\hbar}{2\pi}\kappa$ and the first law of black hole mechanics $\delta E = T_{\mathrm{H}} \delta S_{\mathrm{BH}}$.[1]
- From a saddle point evaluation of the Euclidean partition function.[2]

But the statistical meaning of S_{BH} is not clear.

What is the statistical mechanics of black hole thermodynamics?

[1] Hawking 1975

[2] Gibbons & Hawking 1977

Entanglement entropy

The black hole horizon also has an entanglement entropy.[3]

We have a tensor product decomposition, and partial trace

$$\mathcal{H} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}, \qquad \rho_{\text{out}} = \text{tr}_{\text{in}} |\psi\rangle\langle\psi| \, .$$

The entropy of ρ_Ω is the entanglement entropy:

$$S_{\text{ent}} = -\,\text{tr}\,\rho_{\text{out}} \ln \rho_{\text{out}}.$$

Statistical meaning: entropy comes from missing correlations due to inaccessible black hole interior.

How is this related to the macroscopic quantity S_{BH}?

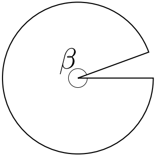
[3] Sorkin 1983; Bombelli, Koul, Lee, & Sorkin 1986; Srednicki 1993

How are S_{BH} and S_{ent} related?

To relate S_{BH} and S_{ent}, we use the conical method.[4]

For simplicity we consider a Rindler horizon.

Let $Z(\beta)$ be the Euclidean path integral on a cone of angle β, times $D-2$ flat and compact transverse directions.



$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

$$S_{\mathrm{cone}} \equiv \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z \bigg|_{\beta = 2\pi}$$

This **conical entropy** is equivalent to varying the period at ∞.

Strategy: Compute S_{cone} in effective theory, and in the quantum theory.

[4] Bañados, Teitelboim & Zanelli 1993

Relating S_{cone} and S_{BH}

Suppose we integrate out the matter fields, leading to an effective action

$$\int \mathcal{D}\phi e^{-S[\phi]} = e^{-\int \sqrt{g} L_{\text{eff}}}, \qquad L_{\text{eff}} = \frac{1}{16\pi G_{\text{eff}}}(R - 2\Lambda_{\text{eff}} + \dots)$$

The conical entropy formula gives the Bekenstein-Hawking formula [5]

$$S_{\text{cone}} = \frac{A}{4G_{\text{eff}}} = S_{\text{BH}}.$$

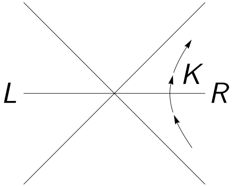
In terms of the **effective** Newton's constant, G_{eff}.

[5] Susskind & Uglum 1994; Jacobson 1994

Relating S_{cone} and S_{ent}

Now consider evaluating the entropy in the quantum theory.

The Minkowski vacuum $|0\rangle$ restricted to one Rindler wedge is thermal in the boost generator K:[6]



$$\rho_R \equiv \operatorname{tr}_L |0\rangle\langle 0| = \frac{e^{-2\pi K}}{Z(2\pi)},$$

$$Z(\beta) \equiv \operatorname{tr} e^{-\beta K}.$$

Varying β is equivalent to varying the temperature of a thermal state:

$$S_{\text{cone}} \equiv \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z \bigg|_{\beta = 2\pi} = -\operatorname{tr} \rho_R \ln \rho_R = S_{\text{ent}}.$$

[6] Bisognano & Wichmann 1975

Not so fast!

The conical geometry also has a singular curvature at the tip:[7]

$$R_{abcd}(x) = (2\pi - \beta)\epsilon_{ab}\epsilon_{cd}\delta_\Sigma(x).$$

Nonminimally coupled matter interacts with this curvature.

The contribution to the conical entropy coming from the tip is:

$$\langle S_{\text{Wald}} \rangle = -2\pi \int_\Sigma \sqrt{h} \left\langle \frac{\partial L}{\partial R_{abcd}} \right\rangle \epsilon_{ab}\epsilon_{cd}.$$

This term is the contribution of the matter fields to the Wald entropy.[8]

Thus for nonminimally coupled matter,[9]

$$S_{\text{BH}} = S_{\text{ent}} + \langle S_{\text{Wald}} \rangle .$$

[7] Fursaev & Solodukhin 1995

[8] Wald 1993; Visser 1993; Jacobson, Kang & Myers 1993

[9] As suggested by arguments in Frolov & Fursaev 1997

One loop results

The conical entropy has been calculated for free fields of spin ≤ 2,

$$S_{\text{cone}} = A\, c_1 \left(2\pi \int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{(4\pi s)^{D/2}} \right).$$

c_1 depends on the field and N, the number of on-shell degrees of freedom[10]

Spin	Field	N	c_1
0	Nonminimally coupled scalar	1	$\frac{N}{6} - \xi$
$\frac{1}{2}$	Dirac spinor	$2^{\lfloor \frac{D}{2} \rfloor - 1}$	$\frac{N}{6}$
1	Maxwell field	$D-2$	$\frac{N}{6} - 1$
$\frac{3}{2}$	Rarita-Schwinger field	$(D-3)2^{\lfloor \frac{D}{2} \rfloor - 1}$	$\frac{N}{6}$
2	Graviton	$\frac{D(D-3)}{2}$	$\frac{N}{6} - \frac{D^2-D+4}{2}$

For gauge fields there is a mysterious contact term.[11]

[10] Solodukhin 2011

[11] Kabat 1995

Electromagnetic field

For the electromagnetic field the Lagrangian is

$$L = \frac{1}{4} F^{ab} F_{ab}, \qquad \Rightarrow \qquad \text{expect} \quad S_{\text{Wald}} = 0.$$

We add ghosts c and $\bar{c}$, a gauge fixing term, and integrate by parts:

$$L' = -\frac{1}{2} A^a (g_{ab} \nabla^2 - R_{ab}) A^b - \bar{c} \nabla^2 c.$$

The Wald entropy contribution from the gauge field is:

$$\langle S_{\text{Wald}} \rangle = -\pi \int_\Sigma \sqrt{h}\, g_\perp^{ab} \langle A_a A_b \rangle .$$

Evaluated using the heat-kernel regularization it gives $c_1 = -1$.

Problem: Gauge invariance? What about $D = 2$, where there are no local degrees of freedom?

Compact spacetime

We now consider $D = 2$ and compactify (e.g. 2D de Sitter).

In 2D, any vector field can be written as

$$A = d\phi + \delta\psi + B, \qquad \Delta B = 0$$

The vector field cancels with the ghosts up to zero modes.

The number of zero modes (vector minus two ghosts) is $2g - 2 = -\chi$, where χ is the Euler characteristic.

Using Gauss-Bonnet, we can write $\chi = \frac{1}{4\pi} \int \sqrt{g} R$.

Zero mode contribution to the effective action is proportional to $\int \sqrt{g} R$

$$S_{\text{zero modes}} = -\left(2\pi \int_{\epsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{(4\pi s)^{D/2}} \right).$$

The $c_1 = -1$ in the conical entropy comes from **zero modes**.

Reduced phase space

Heat kernel method *does not* treat zero modes properly.

2D gauge theory has a huge symmetry group: area-preserving diffeomorphisms. It is "almost topological" and can be solved exactly:[12]

$$Z = \sum_{E \in q\mathbb{Z}} e^{-\frac{1}{2} V E^2}.$$

Since $V \propto \beta$, the conical entropy is

$$S_{\text{cone}} = \left(1 - V \frac{\partial}{\partial V}\right) \ln Z = -\sum_E p_E \ln p_E = S_{\text{ent}}.$$

This is finite, positive, and equal to the entanglement entropy.

One can show that $S_{\text{cone}} = S_{\text{ent}}$ for 2D Yang-Mills as well.

[12] Witten 1991

Conclusion

Conclusions:

- Black hole entropy is closely related to entanglement entropy

$$S_{\mathrm{BH}} = S_{\mathrm{ent}} + \langle S_{\mathrm{Wald}} \rangle \,.$$

- But the “contact term” in the entropy of gauge fields can’t be explained this way; it is absent when the partition function is evaluated carefully (in $D = 2$).

Future work:

- Gauge theories in $D > 2$.
- Linearized gravity.

For details see arXiv:1206.5831.

Thank you.