Quantum corrections to the gravitationally coupled magnetic monopole: residual conformal symmetry and trace anomaly

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Conformally Invariant Action

$$S = \int d^4x \sqrt{-g} \left(C_{\mu\nu\sigma\tau} C^{\mu\nu\sigma\tau} - \frac{1}{4e^2} F^{\,a}_{\mu\nu} F^{\mu\nu}_{a} + D_\mu \phi^a D^\mu \phi_a + \frac{1}{6} R \phi^a \phi_a - \lambda^2 (\phi^a \phi_a)^2 \right)$$

Invariant under: $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$ $\qquad \phi^a \to \Omega^{-1}(x) \phi^a$

Note: no "mass term" $\mu^2 \Phi^2$. Replaced by R Φ^2 term.
-> Spontaneous symmetry breaking (SSB) via gravitation.

VEV: $\phi_0^2 = \dfrac{R}{12\lambda^2}$ AdS background, R=positive constant

It was shown that this allows for a magnetic monopole solution with Schwarzschild-AdS spacetime asymptotically.

SSB does not introduce a length scale

Spontaneous symmetry breaking (SSB) breaks the conformal symmetry but only partially.

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \quad ; \quad \phi_0 \to \frac{\phi_0}{\Omega(x)} \quad ; \quad R \to \frac{R}{\Omega^2(x)} + \frac{6}{\Omega^3(x)} \Box \Omega(x)$$

conformal transformations obeying the condition $\Box \Omega(x) = 0$ leave vacuum invariant.

– dilatations (global scale invariance): $x^\mu \to x'^\mu = \beta x^\mu$

obeys above condition -> no length scale

–special conformal transformations: $x^\mu \to x'^\mu = \dfrac{x^\mu + a^\mu x^2}{1 + 2 a_\beta x^\beta + a^2 x^2}$

obeys condition if $a^2 = a^\mu a_\mu = 0$

* vacuum invariant under 14 parameter subgroup

Quantum corrections introduce a length scale

One loop divergent part of effective action

$$W_{div} = \frac{1}{n-4} \int d^4x \sqrt{g} \;\; \mathrm{tr}\, \hat{a}_2(x,x) \qquad (n \to 4)$$

Schwinger-Dewitt coefficient

$$\hat{a}_2(x,x) = \frac{1}{180}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \Box R\Big)\hat{1} + \frac{1}{12}\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}^{\mu\nu} + \frac{1}{2}\hat{P}^2 + \frac{1}{6}\Box\hat{P}\,.$$

Renormalized constants

$$S_{ren} = \int d^4x \sqrt{-g}\left(\alpha_R R^2 + \beta_R R_{\mu\nu}R^{\mu\nu} - \frac{1}{4e_R^2}F^2 + (D\phi)^2 + \frac{1}{6}R\phi^2 - \lambda_R^2\phi^4\right)$$

running coupling constants governed by an RG equation
-> length scale introduced

Trace anomaly involves composite operators

The SO(2,3) symmetry of AdS background yields

$$< T_{\mu\nu} > \equiv \frac{2}{\sqrt{-g}} \frac{\delta W_{ren}}{\delta g^{\mu\nu}} = \frac{1}{4} g_{\mu\nu} < T^{\mu}_{\mu} >$$

$$< T^{\mu}_{\mu} > = \frac{3\lambda^2}{16\,\pi^2}[\phi^4] - [E] + \frac{1}{16\pi^2}\Big[\frac{1}{60}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \Box R\Big) - \frac{1}{6}F^2 - \frac{10}{3}\lambda^2 \Box [\phi^2]\Big]$$

$$[E] \equiv \frac{\phi^a}{\sqrt{-g}} \frac{\delta W_{ren}}{\delta \phi^a}$$

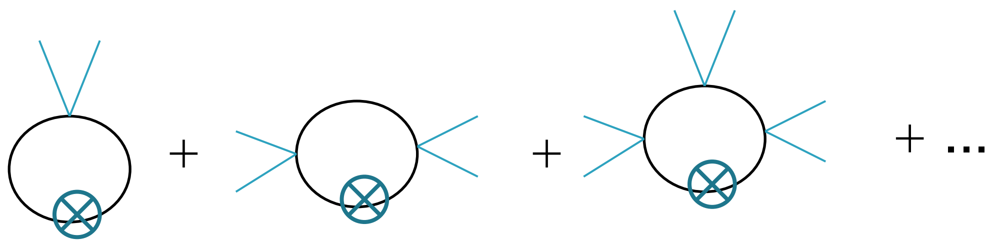
[ ]=composite operators

need to calculate value of composite operators in vacuum
with spontaneous symmetry breaking (non-zero VEV Φ_0)

use the effective potential formalism

One loop effective potential U with composite operator insertion: [Φ²] example

[Φ²]: insert an extra vertex $\otimes$ in one loop Feynman diagrams with zero external momenta.



$$U(\phi_0) = \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^n(0; 0, 0, 0, ...0) \phi_o(x)^n$$

Tree level (zero loop): $U_0 = \phi_0^2$

one loop and renormalization

In AdS, R=constant=12k. Define μ^2 =R/3=4k.

$$\Gamma^{2n}(0;0,..0) = \frac{(2n)!}{2\,2^n}\int \frac{d^4k}{(2\pi)^4}\left((-i\lambda)\frac{i}{k^2-\mu^2+i\epsilon}\right)^n \frac{i}{k^2-\mu^2+i\epsilon}$$

$$\begin{aligned}U_1 &= \frac{-1}{2}\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+\mu^2-i\epsilon}\,\frac{\lambda\phi_0^2}{2(k_E^2+\mu^2+\lambda\phi_c^2/2-i\epsilon)} \\ &= \frac{-\lambda\phi_o^2}{16\pi^2}\int_0^\Lambda \frac{k_E^3\,dk}{(k_E^2+\mu^2+\lambda\phi_0^2/2)\,(k_E^2+\mu^2)} \\ &= \frac{-1}{32\pi^2}\left((2\mu^2+\lambda\phi^2)\,\ln\left(\tfrac{2\,\Lambda^2+2\,\mu^2+\lambda\,\phi_o^2}{2\mu^2+\lambda\phi_0^2}\right) - 2\mu^2\ln(\Lambda^2+\mu^2) + 2\mu^2\ln(\mu^2)\right)\end{aligned}$$

Add counterterm $U_{ct} = A\phi_0^2$. The constant A is determined by the renormalization condition $\Gamma^2(0;0,0)=2$ which implies $\frac{d^2U}{d\phi_0^2}|_{\phi_0=0} = 2$

$$U(\phi_0) = \phi_0^2 + \frac{\mu^2}{16\pi^2}\ln\left(\tfrac{\mu^2+\lambda\phi_0^2}{\mu^2}\right) - \frac{1}{32\pi^2}\,\frac{2\lambda\mu^2\phi_0^2+3\lambda^2\phi_0^4}{2\mu^2+\lambda\phi_0^2}$$

Commutator Curvature

$$[D_\mu, D_\nu]\,\phi^{\,a} = \mathscr{R}^{\,a}{}_{b\,\mu\,\nu}\,\phi^{\,b} \text{ with } \hat{\mathscr{R}}_{\mu\nu} \equiv \mathscr{R}^{\,a}{}_{b\,\mu\,\nu}$$

$$\begin{aligned}
D_\mu D_\nu \phi^{\,a} &= \nabla_\mu (D_\nu \phi^{\,a}) + \varepsilon^a{}_{bc}\,A^b_\mu\,D_\nu \phi^{\,c} \\
&= \nabla_\mu (\nabla_\nu \phi^{\,a} + \varepsilon^a{}_{de}\,A^d_\nu\,\phi^{\,e}) + \varepsilon^a{}_{bc}\,A^b_\mu (\nabla_\nu \phi^{\,c} + \varepsilon^c{}_{fg}\,A^f_\nu\,\phi^{\,g}) \\
&= \nabla_\mu \nabla_\nu \phi^a + \varepsilon^a{}_{de}\,\nabla_\mu A^d_\nu\,\phi^{\,e} + \varepsilon^a{}_{de}\,A^d_\nu \nabla_\mu \phi^{\,e} + \varepsilon^a{}_{bc}\,A^b_\mu \nabla_\nu \phi^{\,c} + \varepsilon^a{}_{bc}\,\varepsilon^c{}_{fg}\,A^b_\mu\,A^f_\nu\,\phi^{\,g}\,.
\end{aligned}$$

The commutator is then given by

$$\begin{aligned}
[D_\mu, D_\nu]\phi^{\,a} &= \varepsilon^a{}_{de}\left(\nabla_\mu A^d_\nu - \nabla_\nu A^d_\mu\right)\phi^{\,e} + \varepsilon^a{}_{bc}\,\varepsilon^c{}_{fg}\left(A^b_\mu\,A^f_\nu - A^b_\nu\,A^f_\mu\right)\phi^{\,g} \\
&= \varepsilon^a{}_{de}\left(\nabla_\mu A^d_\nu - \nabla_\nu A^d_\mu + \varepsilon^d{}_{fg}\,A^f_\mu\,A^g_\nu\right)\phi^{\,e} \\
&= \varepsilon^a{}_{de}\,F^{\,d}_{\mu\nu}\,\phi^{e}
\end{aligned}$$

$$\hat{\mathscr{R}}_{\mu\nu} \equiv \mathscr{R}^{\,a}{}_{e\,\mu\,\nu} = \varepsilon^a{}_{de}\,F^{\,d}_{\mu\nu}$$

$$P_{ij} = -\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}\lambda^2\left(\phi_a\phi^a\right)^2 = -4\,\lambda^2\left(\delta_{ij}\,\phi_a\phi^a + 2\,\phi_i\,\phi_j\,\right)$$

Notation

We use the notation of Mukhanov & Winitzki, *Introduction to Quantum Effects in Gravity* (2007).

Metric signature is (+,-,-,-),

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \ldots \text{ and } R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$$