Averaged null energy condition in curved space

Eleni-Alexandra Kontou and Ken D. Olum

Institute of Cosmology, Department of Physics and Astronomy
Tufts University

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Let $M$ a manifold, $g$ its lorentzian metric. Also let $\gamma$ an achronal null geodesic and $l^a$ its tangent vector. Then

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⇒ Achronal ANEC was proved (Fewster, Olum, Pfenning. 2007) to hold for geodesics in curved space, providing that any curvature stays some minimum distance from the geodesic, which then are travelling in flat space. Here we will try to prove it for geodesics travelling in curved space.
Assumptions

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2 **Coordinate system:** Fermi-like coordinates.

3 **Curvature:** We require that $|R_{\mu\nu\rho\sigma}| < R_{\text{max}}$ everywhere in $M'$. We also require the null convergence condition $R_{ab}l^a l^b \geq 0$ for any null vector $l$ which holds whenever the curvature is generated by a classical background whose stress tensor obeys the Null Energy Condition (NEC).
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4 **Causal structure**: Conditions outside $M'$ do not affect the causal structure of $M'$

\[ J^+(p, M) \cap M' = J^+(p, M') \]
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$$J^+(p, M) \cap M' = J^+(p, M')$$

5 **Quantum field theory**: We consider a quantum scalar field in $M$, which inside $M'$ is free and minimally coupled.
Quantum inequality

In flat spacetime it was proved (Fewster, Roman 2002) that

\[
\int_{-\tau_0}^{\tau_0} d\tau \ T_{ab}(w(\tau)) l^a l^b f(\tau/\tau_0)^2 \geq - \frac{(k_a l^a)^2}{12\pi^2 \tau_0^4} \int_{-\tau_0}^{\tau_0} d\tau f''(\tau/\tau_0)^2
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where \( f \) is compact function with \( \int_{-1}^{1} dx f(x)^2 = 1 \).
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where \( f \) is compact function with \( \int_{-1}^{1} dx f(x)^2 = 1 \). Suppose we want to test quantum inequality in a laboratory on the surface of the earth... This is not flat space but it has curvature of order \( GM_0 / R_0^3 \).

⇒ We expect QI to hold with a small correction in globally hyperbolic spacetimes with small curvature: \( |R_{abcd}|\tau_0^2 < \epsilon \), where \( \epsilon \ll 1 \) and small proper acceleration of the timelike paths.
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Conjecture

$$\int_{-\tau_0}^{\tau_0} d\tau \, T_{ab}(w(\tau)) l^a l^b f(\tau/\tau_0)^2 \geq -\left(\frac{k_a l^a}{12\pi^2 \tau_0^4}\right) \int_{-\tau_0}^{\tau_0} d\tau f''(\tau/\tau_0)^2 [1 + c\epsilon]$$
Proof of achronal ANEC

1 The parallelogram

Consider the points \( \Phi(u, v) = (u, v, 0, 0) \), null geodesics in \( M' \). The ANEC integral can be written as

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A(v) = \int_{-\infty}^{\infty} du T_{uu}(\Phi(u, v))
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Define \( \tau_0 = \gamma^{-\alpha} r \) where \( 0 < \alpha < 1/3 \) and \( r \) a positive number with dimensions of length. As \( V \to 1, \gamma \to \infty \) and \( \tau_0 \to 0 \). Now consider the points \( \Phi_V(\eta, \tau) = \Phi(u, v) \) we can write the ANEC integral as
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$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau T_{uu}(\Phi_V(\eta, \tau)) f(\tau/\tau_0)^2 < -A\tau_0$$
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Timelike paths

\[ -\tau_0 < \tau < \tau_0 \]

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Null paths

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\[ |R_{a'b'c'd'}| < R_{\text{max}} \]
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  $g_{\alpha'\beta'} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'} = \eta_{\alpha'\beta'} + O(RX^2)$ where $X$ denotes coordinate values.

We apply the QNEI in the globally hyperbolic causal diamond $N = J^+(p) \cap J^-(q)$ which we proved that is inside the tube after the boost.
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- In our case we can easily prove that $h_{\alpha'\beta'} = O(R_{\max}\tau_0^2)$ so
  
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- We apply the QNEI in the globally hyperbolic causal diamond
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3 Quantum Inequality

We showed that the curvature is bounded, paths $\Phi_V$ are timelike and proper acceleration is small. So now we can apply the quantum inequality for small curvature:

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau T_{uu}(\Phi_V(\eta, \tau)) f(\tau/\tau_0)^2 \geq -\frac{F\eta_0}{12\pi^2\gamma^2\tau_0^3} \left[1 + O(R_{max}\tau_0^2)\right]$$

Where we used $(l_\alpha k^\alpha)^2 \sim 1/\gamma^2$ and $F = \tau_0^{-1} \int d\tau f''(\tau/\tau_0)^2$. 
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Where we used $(l_\alpha k^\alpha)^2 \sim 1/\gamma^2$ and $F = \tau_0^{-1} \int d\tau f''(\tau/\tau_0)^2$. The right hand of this equation goes like $\frac{\eta_0}{\gamma^2\tau_0^3} \sim \gamma^{2\alpha-1}$ while in the ANEC inequality

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau \ T_{uu}(\Phi_V(\eta, \tau)) f(\tau/\tau_0)^2 < -A\tau_0$$

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$\Rightarrow$ Since $\alpha < 1/3$ the lower bound in the first equation goes to zero faster than the upper bound in the second equation. This contradiction proves the theorem.
Conclusions

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