Topological effects in linear gauge theories

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Joint work (in preparation) with Claudio Dappiaggi (Pavia) and Thomas-Paul Hack (Hamburg).
Recently, Dappiaggi et al. considered the free (quantum) vector potential of electromagnetism in curved spacetimes, in the light of general covariance. Their conclusions:

- The Poisson bracket may be degenerate, depending on the topology of the background spacetime.
- The theory shows non-local effects: the degenerate observables may vanish, after embedding a spacetime into a larger one.

The interpretation of these degenerate observables remained to be clarified.
In this talk we use the vector potential to illustrate a general formalism that helps to clarify these issues. We address the following questions:

- How do we compute the degeneracies of the Poisson bracket?
- How do we interpret these degeneracies?

For the vector potential this leads to an apparently new insight in the relation between the Aharonov-Bohm effect and Gauss’ law.
Electromagnetism in Minkowski spacetime

In Minkowski spacetime $M_0$, electromagnetism is described by:

- A Maxwell field $F \in \Omega^2(M_0)$ such that
  \[ dF = \nabla[\mu F_{\nu\rho}] = 0 \quad \delta F = \nabla^\mu F_{\mu\nu} = J = 0, \]

- or a vector potential $A \in \Omega^1(M_0)$ such that $(F = dA)$
  \[ \delta dA = J = 0 \quad A \sim 0 \Longleftrightarrow dA = 0, \]

- or a vector potential $A \in \Omega^1(M_0)$ such that
  \[ \delta dA = J = 0 \quad A \sim 0 \Longleftrightarrow A = d\chi, \, \chi \in \Omega^0(M_0). \]

In general spacetimes these formulations are no longer equivalent!
The Aharonov-Bohm effect and the gauge symmetry

The Aharonov-Bohm effect allows us to distinguish between $A_1$ and $A_2$ when $A_1 - A_2$ is not exact:

$$A = d\phi,$$

in a region outside a coil, is closed, but not exact. Using quantum particles one can measure a phase shift $\sim \oint A$.

We therefore consider the theory for $A \in \Omega^1(M)$ such that

$$\delta dA = 0 \quad A \sim 0 \iff A = d\chi, \; \chi \in \Omega^0(M).$$
Local observables

If $\Sigma \subset M$ is a Cauchy surface, the space of field configurations is parametrised by initial data:

$$\mathcal{F} = \left\{ E \in \Omega^1(\Sigma) \mid \delta E = 0 \right\} \oplus \left\{ a \in \Omega^1(\Sigma) \right\} / d\Omega^0(\Sigma).$$

A local, linear observable is given by

$$\langle (\alpha, \epsilon), (E, a) \rangle := \int_\Sigma \epsilon \wedge * a - \alpha \wedge * E = \int_\Sigma \epsilon_\mu a^\mu - \alpha_\mu E^\mu,$$

with $(\alpha, \epsilon)$ in the dual space

$$\mathcal{F}' = \left\{ \alpha \in \Omega^1_0(\Sigma) \right\} / d\Omega^0_0(\Sigma) \oplus \left\{ \epsilon \in \Omega^1_0(\Sigma) \mid \delta \epsilon = 0 \right\}.$$

The pairing $\langle \ , \rangle : \mathcal{F}' \times \mathcal{F} \to \mathbb{C}$ is non-degenerate in both entries.
Peierls’ Poisson bracket

The Poisson bracket on $\mathcal{F}'$ can be obtained from the Lagrangian of the theory by a general procedure due to Peierls (1952). It yields:

$$\{(\alpha_1, \epsilon_1), (\alpha_2, \epsilon_2)\} = \int_\Sigma \epsilon_1 \wedge * \alpha_2 - \alpha_1 \wedge * \epsilon_2.$$ 

Remarks:

- The Poisson bracket is an important structure e.g. for canonical quantisation (or deformation quantisation).
- The Poisson bracket is an anti-symmetric linear map on (linear) observables in $\mathcal{F}'$. (The symplectic form, on the other hand, is a map on $\mathcal{F}$.)
Degeneracies of the Poisson structure

The Poisson bracket is (in general) degenerate:

\[ \{ (\alpha, \epsilon), (\alpha', \epsilon') \} = 0 \quad \forall (\alpha', \epsilon') \in \mathcal{F}' \]

\[ \iff \]

\[ \epsilon = 0, \quad \alpha \in \text{deg}(\Sigma) := \left( \Omega_0^1(\Sigma) \cap d\Omega_0^0(\Sigma) \right) / d\Omega_0^0(\Sigma). \]

I.e. $\alpha = d\beta$, $\alpha$ has compact support, but $\beta$ does not.

Question:
What do the degenerate observables measure? The Aharonov-Bohm effect?
Consider an ultrastatic spacetime with $\Sigma := \mathbb{R}^3 \setminus \{0\} = \mathbb{R}_{>0} \times S^2$. Let $\beta \in \Omega^0(\Sigma)$ be

- rotation invariant,
- $\equiv 1$ on $r \leq R$,
- $\equiv 0$ on $r \geq R + \epsilon$,

where $r$ is a radial coordinate. Then $\alpha := d\beta = \beta'(r)dr \in \deg(\Sigma)$.

The observable $(\alpha, 0)$ measures (a multiple of) the electromagnetic flux through the shell $1 \leq r \leq 2$. 
Gauss’ law

All degenerate observables are of this type: they use Gauss’ law to measure electric charges which lie outside the spacetime itself.

The (possible) electric charges of a spacetime are characterised by the possible degenerate observables, i.e. by

$$\deg(\Sigma) = \left( \Omega^1_0(\Sigma) \cap d\Omega^0(\Sigma) \right) / d\Omega^0_0(\Sigma).$$

When $\Sigma$ is compact, $\deg(\Sigma) = \{0\}$.
When $H^1(\Sigma) = \{0\}$, $\deg(\Sigma) = H^1_0(\Sigma)$. A basis of degenerate observables is then indexed by non-contractible spheres in $\Sigma$, up to homology.
In general, a basis of degenerate observables is indexed by non-contractible spheres in $\Sigma$, up to homology, which cut $\Sigma$ into two non-compact pieces.

A pedagogical example is $\Sigma := S^1 \times S^2$. Here $H_0^1(\Sigma) \cong \mathbb{R}$, but $\text{deg}(\Sigma) = \{0\}$ as $\Sigma$ is compact.

Physical intuition:
Removing any non-contractible sphere from $\Sigma$ leaves a single connected set. The sphere does not separate a point charge from infinity. There is no charge.
The Aharonov-Bohm effect motivated the choice of gauge equivalence.

By general procedures we found the Poisson structure and its degeneracies.

The degeneracies correspond to Gauss’ law and yield a topological formula for electric monopoles.

The same mathematical argument works for $p$-form fields and magnetic monopoles, also when source currents are present.

The same argument should apply to other linearised gauge theories (e.g. linearised GR).
The space of field configurations is

\[ \mathcal{F} := \left\{ A \in \Omega^1(M) \mid \delta dA = 0 \right\} / d\Omega^0(M). \]

A local, linear observable is

\[ f(A) := \langle f, A \rangle := \int_M f \wedge \ast A , \quad f \in \Omega^1_0(M). \]

The space of such observables is

\[ \mathcal{F}' := \left\{ f \in \Omega^1_0(M) \mid \delta f = 0 \right\} / \delta d\Omega^1_0(M) \]

so that the pairing

\[ \mathcal{F}' \times \mathcal{F} \ni (f, A) \mapsto \langle f, A \rangle \]

is non-degenerate in both entries.
The Poisson bracket is

$$\{ f_1, f_2 \} = \int_M f_1 E f_2,$$

where $E$ is the advanced-minus-retarded fundamental solution of a hyperbolic (Laplace-Beltrami) operator obtained by fixing a Lorenz gauge.

The space of degenerate observables is

$$\{ f, f' \} = 0 \quad \forall f' \in \mathcal{F}' \quad \Leftrightarrow \quad f \in \Omega_0^1(M) \cap \delta d\Omega_{tc}^1(M),$$

where $tc$ means time-like compact support.
Poisson brackets vs. symplectic forms

We may view $\mathcal{F}$ as an infinite dimensional manifold. Then,

$$TF \simeq \mathcal{F} \times \mathcal{F}, \quad T^*\mathcal{F} \simeq \mathcal{F} \times \mathcal{F}'.$$

The Poisson bracket is a two-vector field $P$:

$$\{f_1, f_2\} = P^{ab} (f_1)_a (f_2)_b, \quad f_1, f_2 \in T^*_A \mathcal{F} \simeq \mathcal{F}'.$$

One may also consider a symplectic form (up to technicalities)

$$\Omega(\delta_1 A, \delta_2 A) = \Omega_{ab} (\delta_1 A)^a (\delta_1 A)^b, \quad \delta_1 A, \delta_2 A \in T_A \mathcal{F} \simeq F.$$

(See e.g. Lee and Wald (1990).)

In finite dimensions and without degeneracies, $P^{ab}$ and $\Omega_{ab}$ are each other’s inverses. In general, the situation is not so clear.