# 𝒞ausal structure of black hole interiors in spherical symmetry


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# 𝒪verview

Why study black hole interiors? Our motivation is three-fold:

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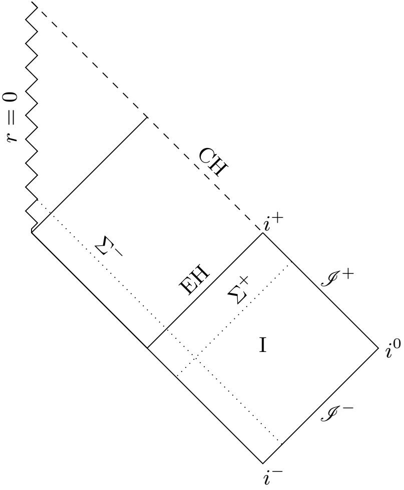
(1) Our goal is to investigate numerically the singularity inside a more realistic rotating black hole, to understand how gravitational collapse affects the causal structure beneath the event horizon (EH).

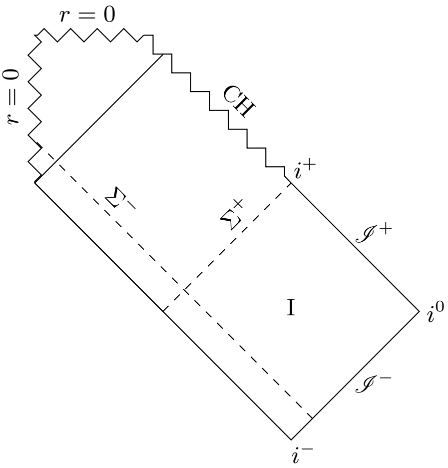
(2) That problem is very difficult, so here I report on an attempt to "get my feet wet" by numerically integrating Einstein field equations for a spherically symmetric, charged black hole perturbed by a scalar field. This reproduces earlier results of *e.g.* Brady & Smith (1995).

(3) Finally, I expand on this work by studying the tidal deformation along timelike geodesics that intersect the Cauchy horizon (CH), and comment on a recent result of Marolf & Ori.

# $\mathscr{I}$nitial Data Structure

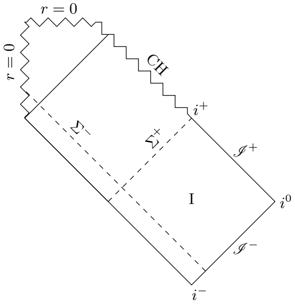


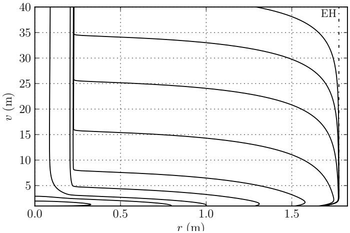

Exact Reissner-Nördstrom




R-N + scalar field

# $\mathscr{M}$odel of the Interior




- Coordinate choice: We use a half-null chart with line element

$$ds^2 = -\frac{\eta\Phi}{r}\,dv^2 + 2\eta\,dr\,dv + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi\right)$$

  where $(\eta, \Phi)$ are the available gravitational degrees of freedom, since it covers the entire region of interest in the figure at left

- Field equations: The spacetime evolves according to

$$(r\psi)_{,r} = \bar{\psi} \qquad r_{,v} = \Phi/2r$$

$$\Phi_{,r} = \eta\left(1 - q^2/r^2\right) \qquad (\ln\eta)_{,r} = r^{-1}\left(\psi - \bar{\psi}\right)^2$$

$$\bar{\psi}_{,v} + (\Phi/2r)\,\bar{\psi}_{,r} = \frac{\psi - \bar{\psi}}{2r}\left[\eta\left(1 - q^2/r^2\right) - \Phi/r\right]$$

- Flux of scalar matter $\mathscr{F} \propto (d\psi/dv)^2$ across the EH obeys Price's law, representing late-time decay of the radiative tail of gravitational collapse
- Scattering of $\psi$ off the interior induces contraction of the null generators of the CH, where buildup of outgoing radiation diverges in the limit $v \to \infty$
- There is a thick (in terms of $r$) layer over which linear perturbation theory remains valid

# $\mathscr{T}$he Late-time Linear Regime

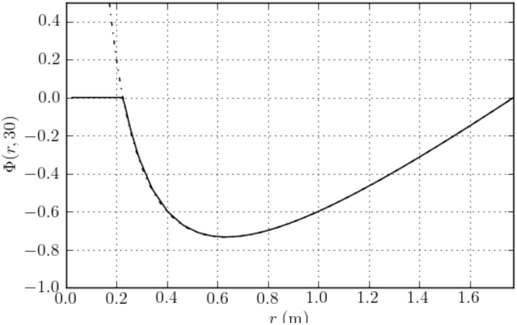



FIG. 3: Behavior of the metric function $\Phi(r, v)$ at fixed $v \gg v_0$. Dashed curve indicates the exact value $\Phi_{RN} = r - 2 + 0.4/r$.

# $\mathscr{R}$adial Observers

But: Our model is not static. There is no conserved energy for particle motion!

- Consider the single-particle Lagrangian

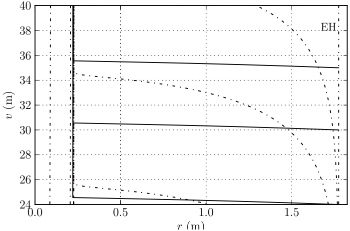
$$\mathscr{L} = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b = -\frac{\eta\Phi}{2r}\dot{v}^2 + \eta\dot{r}\dot{v}$$

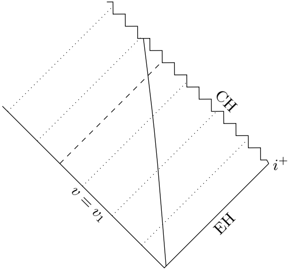
for radial motion and let $\xi = -\eta\dot{v}$. Crucially, **it is still possible to write down first integrals of the motion.**

- Timelike geodesics: From the constraint $\mathscr{L} = -1/2$, the definition of $\xi$, and the form of variations in $r$, we obtain the first-order system

$$\begin{aligned}
\dot{v} &= -\eta^{-1}\xi \\
\dot{r} &= \frac{\xi}{2}\left(\xi^{-2} - \frac{1}{\eta}\frac{\Phi}{r}\right) \\
\dot{\xi} &= \frac{1}{2r}\left[\xi^2\left(1 - \frac{q^2}{r^2} - \frac{1}{\eta}\frac{\Phi}{r}\right) + (\psi - \bar{\psi})\right]
\end{aligned}$$

- Successfully tested in exact R-N with a symplectic integrator (necessary because the evolution equations are stiff)
- Geodesics plotted at right each begin on the EH at some advanced time $v_1 \gg v_0$ with $\xi|_{\mathrm{EH}} = -1/2$. Corresponding curves at earlier infall times have markedly different behavior.



# $\mathscr{R}$adial Observers

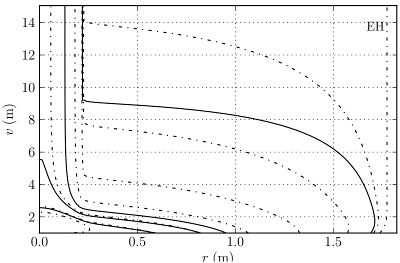
A Jacobi field is a solution to the geodesic deviation equation, $D^2\eta^a + R^a{}_{bcd}\dot{x}^b\eta^c\dot{x}^d = 0$, where $D = \dot{x}^a\nabla_a$. These give information about the tidal deformation of a material body whose center of mass moves along some timelike geodesic curve $\gamma(\tau)$.

- The spacelike Jacobi fields are spanned by a mutually orthogonal triad,

$$\boldsymbol{\eta}_1 = a\sqrt{\left(\frac{\eta\Phi}{r} + \frac{\dot{r}}{\dot{v}}\right)^{-1}}\left[\frac{\partial}{\partial v} + \left(\frac{\Phi}{r} - \frac{\dot{r}}{\dot{v}}\right)\frac{\partial}{\partial r}\right]$$

$$\boldsymbol{\eta}_2 = x\frac{\partial}{\partial\theta} \qquad \boldsymbol{\eta}_3 = y\csc\theta\,\frac{\partial}{\partial\varphi}$$

- Thus $\|\boldsymbol{\eta}_1\| = |a|$, $\|\boldsymbol{\eta}_2\| = r|x|$, and $\|\boldsymbol{\eta}_3\| = r|y|$



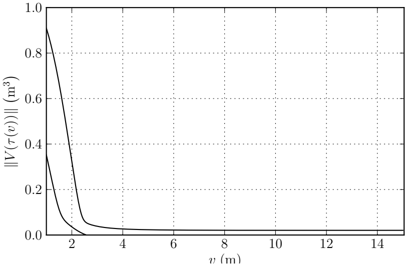

- These quantities have evolution equations

$$r\ddot{x} + 2\dot{r}\dot{x} = 0 \qquad r\ddot{y} + 2\dot{r}\dot{y} = 0$$

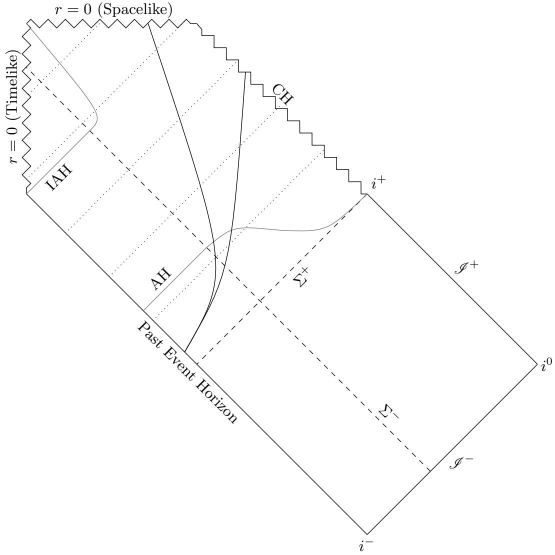
$$\ddot{a} + a\left(2r^{-3}M + 4\Psi_2 - R/6\right) = 0$$

where $M$ is the Misner-Sharp mass, $\Psi_2$ a Newman-Penrose Weyl scalar, and $R$ the Ricci curvature scalar

- Finally, a volume is specified by $\|V(\tau)\| = |axy|r^2$



# $\mathscr{C}$onclusion



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