Center for Particle Cosmology at the University of Pennsylvania

Ward Identities in Cosmology

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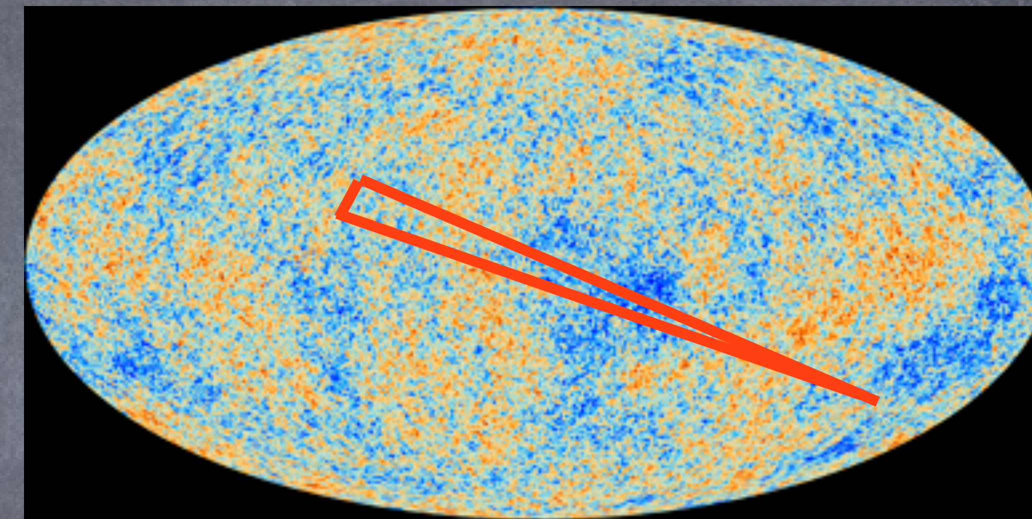
Hinterbichler, Hui & JK, 1203.6351, 1304.5527
Berezhiani & JK, 1309.4461, 1406.2689
Berezhiani, JK & Wang, 1401.7991

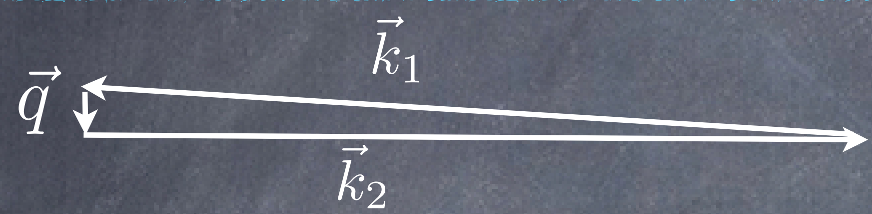
Related work:

Creminelli, Norena & Simonovic, 1203.4595
Goldberger, Hui & Nicolis, 1303.1193
Assasi, Baumann & Green, 1204.4207
Collins, Holman & Vardanyan, 1405.0017
Dimastrogiovanni, Fasiello, Jeong & Kamionkowski, 1407.8204
Armendariz-Picon, Neelakanta & Penco, to appear

Single-field consistency relations



$$\lim_{\vec{q}\to 0} \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle}{P_\zeta(q)} = -(n_s - 1) P_\zeta(k_1)$$



Maldacena (2002); Creminelli & Zaldarriaga (2004); Cheung, Fitzpatrick, Kaplan & Senatore (2007).

- Holds in all inflationary models, under the assumptions:
  - single "clock"
  - Bunch-Davies vacuum (necessary?)
  - background is attractor $\zeta \to \text{const.}$
- Measuring (primordial) 3-point function in this limit

  $\Longrightarrow$ automatically rules out all standard single-field models
  - Planck: $f_{\rm NL}^{\rm local} = 2.7 \pm 5.8$
- Consequence of symmetry: Ward identity for dilation

Background wave

Maldacena (2002); Creminelli & Zaldarriaga (2004)

$$\langle \zeta_S \zeta_S \rangle_{\zeta_L} = \langle \zeta_S \zeta_S \rangle_0 + \zeta_L \frac{\mathrm{d}}{\mathrm{d}\zeta_L} \langle \zeta_S \zeta_S \rangle \Big|_0$$

$$= \langle \zeta_S \zeta_S \rangle_0 + \zeta_L \frac{\mathrm{d}}{\mathrm{d}\ln|\vec{x}_1 - \vec{x}_2|} \langle \zeta_S \zeta_S \rangle \Big|_0$$

Multiply by ζ_L and take expectation value:

$$\langle \zeta_L \langle \zeta_S \zeta_S \rangle_{\zeta_L} \rangle = \langle \zeta_L \zeta_L \rangle \frac{\mathrm{d}}{\mathrm{d}\ln|\vec{x}_1 - \vec{x}_2|} \langle \zeta_S \zeta_S \rangle$$

$$\Longrightarrow \boxed{\lim_{\vec{q}\to 0} \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle}{P_\zeta(q)} = -(n_s - 1) P_\zeta(k_1)}$$

"Background wave" argument is intuitive and compelling, but...

- Semi-classical
- Technically challenging for other symmetries
- Dependence on initial state unclear

The upshot of field theoretic method:

- Non-perturbative
- Easily generalizes to other symmetries
- Dependence on initial state is explicit

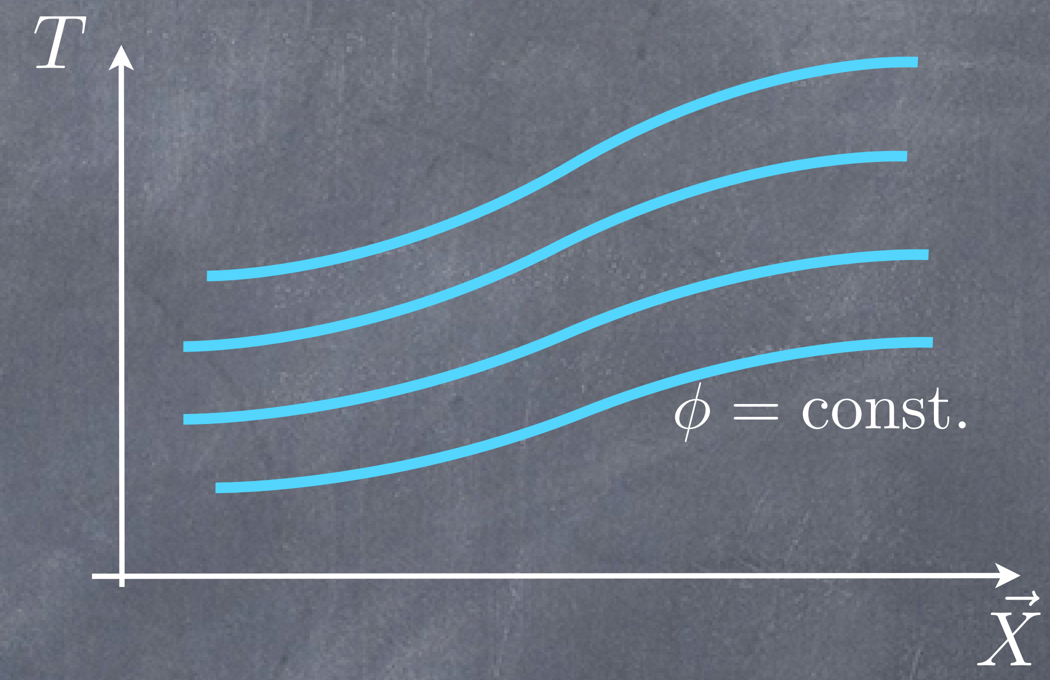
Conformal Symmetries of Scalars

Creminelli, Norena & Simonovic, 1203.4595; Hinterbichler, Hui & Khoury, 1203.6351

Uniform-density gauge:

$$\phi = \phi(t)\,;$$
$$h_{ij} = a^2(t)e^{2\zeta(t,\vec{x})}\delta_{ij}$$

Bardeen, Steinhardt & Turner (1982);
Bond & Salopek (1990)



This completely fixes the gauge, as long as we restrict to diffs that fall off at infinity. $\Longrightarrow$ Focus on diffs that do not fall off.

e.g. Spatial dilation:

$$\vec{x} \to e^{\lambda}\vec{x}$$
$$\zeta \to \zeta + \lambda$$

leaves h_{ij} invariant.

More generally, $h_{ij} = a^2(t)e^{2\zeta(t,\vec{x})}\delta_{ij}$ is preserved by

Conformal transf'n: $\delta_{ij} \to e^{2\Omega(x)}\delta_{ij}$ + Shift: $\zeta \to \zeta + \Omega$

Conformal transf'ns on R^3 form the group $SO(4,1)$:

- Rotations + Translations $\delta\zeta = 0$ — Unbroken (linearly realized)
- Dilation $x^i \to (1+\lambda)x^i$, $\delta\zeta = \lambda$
- Special conformal transformations (SCTs) $x^i \to x^i + 2\,\vec{x}\cdot\vec{b}\,x^i - b^i\vec{x}^2$, $\delta\zeta = -2\vec{b}\cdot\vec{x}$

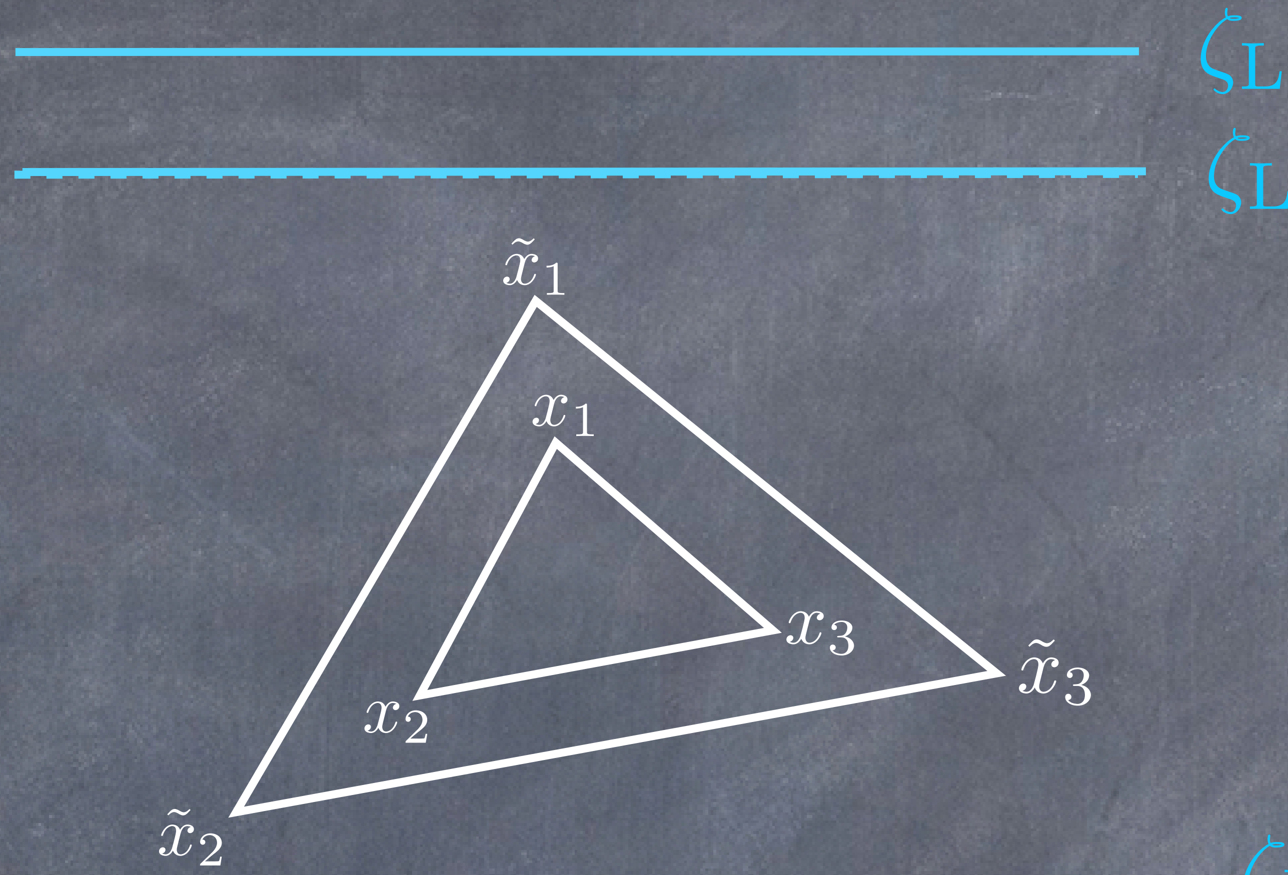
Dilation and SCTs: Spontaneously broken (non-linearly realized)

$\therefore$ $$so(4,1) \to \text{rotations} + \text{translations}$$

ζ is Goldstone boson (dilaton) for the broken symmetries

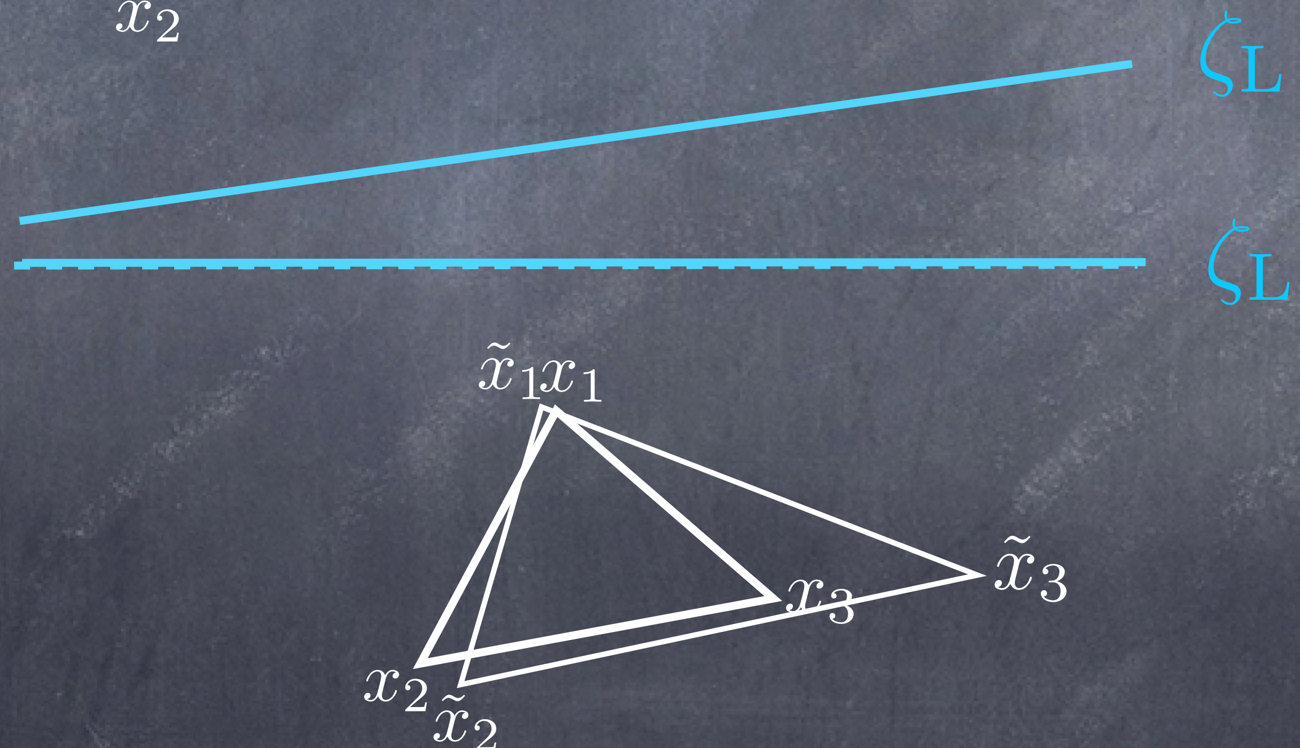
Inf'n = spontaneously broken dS

- Dilation



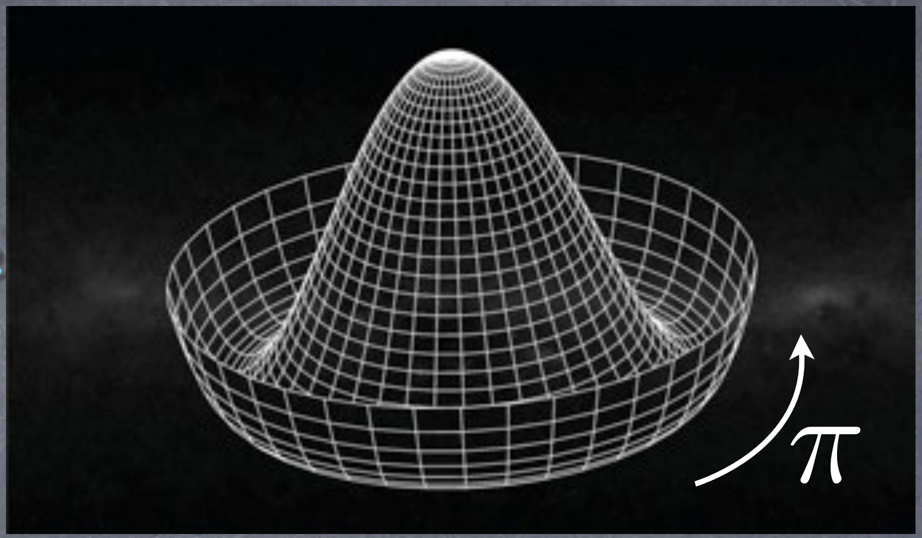
$$\delta\zeta = \lambda$$

- Special conf.



$$\delta\zeta = -2\vec{b}\cdot\vec{x}$$

Ward identities for broken symmetries



Homogeneous Goldstone π is equivalent to change of the vacuum, i.e. to a broken symmetry transformation.

Soft pion thms: $$\lim_{\vec{q}\to 0}\langle \pi(\vec{q})\mathcal{O}(\vec{k}_1,\ldots,\vec{k}_N)\rangle \sim \langle \delta\mathcal{O}(\vec{k}_1,\ldots,\vec{k}_N)\rangle$$

e.g. Strong interactions

Consistency relations as Ward identities

Assasi, Baumann and Green, 1204.4207
Hinterbichler, Hui and Khoury, 1304.5527
Goldberger, Hui and Nicolis, 1303.1193

Dilation:

$$\lim_{\vec{q}\to 0}\frac{1}{P_\zeta(q)}\langle \zeta(\vec{q})\mathcal{O}^\zeta(\vec{k}_1,\ldots,\vec{k}_N)\rangle'_c = -\left(3(N-1)+\sum_{a=1}^{N}\vec{k}_a\cdot\frac{\partial}{\partial\vec{k}_a}\right)\langle \mathcal{O}^\zeta(\vec{k}_1,\ldots,\vec{k}_N)\rangle'_c$$

Special conformal:

$$\lim_{\vec{q}\to 0}\frac{\partial}{\partial q^i}\left(\frac{1}{P_\zeta(q)}\langle \zeta(\vec{q})\mathcal{O}^\zeta(\vec{k}_1,\ldots,\vec{k}_N)\rangle'_c\right) = -\frac{1}{2}\sum_{a=1}^{N}\left(6\frac{\partial}{\partial k_a^i} - k_a^i\frac{\partial^2}{\partial k_a^j\partial k_a^j} + 2k_a^j\frac{\partial^2}{\partial k_a^j\partial k_a^i}\right)\langle \mathcal{O}^\zeta(\vec{k}_1,\ldots,\vec{k}_N)\rangle'_c$$

Creminelli, Norena & Simonovic, 1203.4595

General Ward Identities

Hinterbichler, Hui and Khoury, 1304.5527

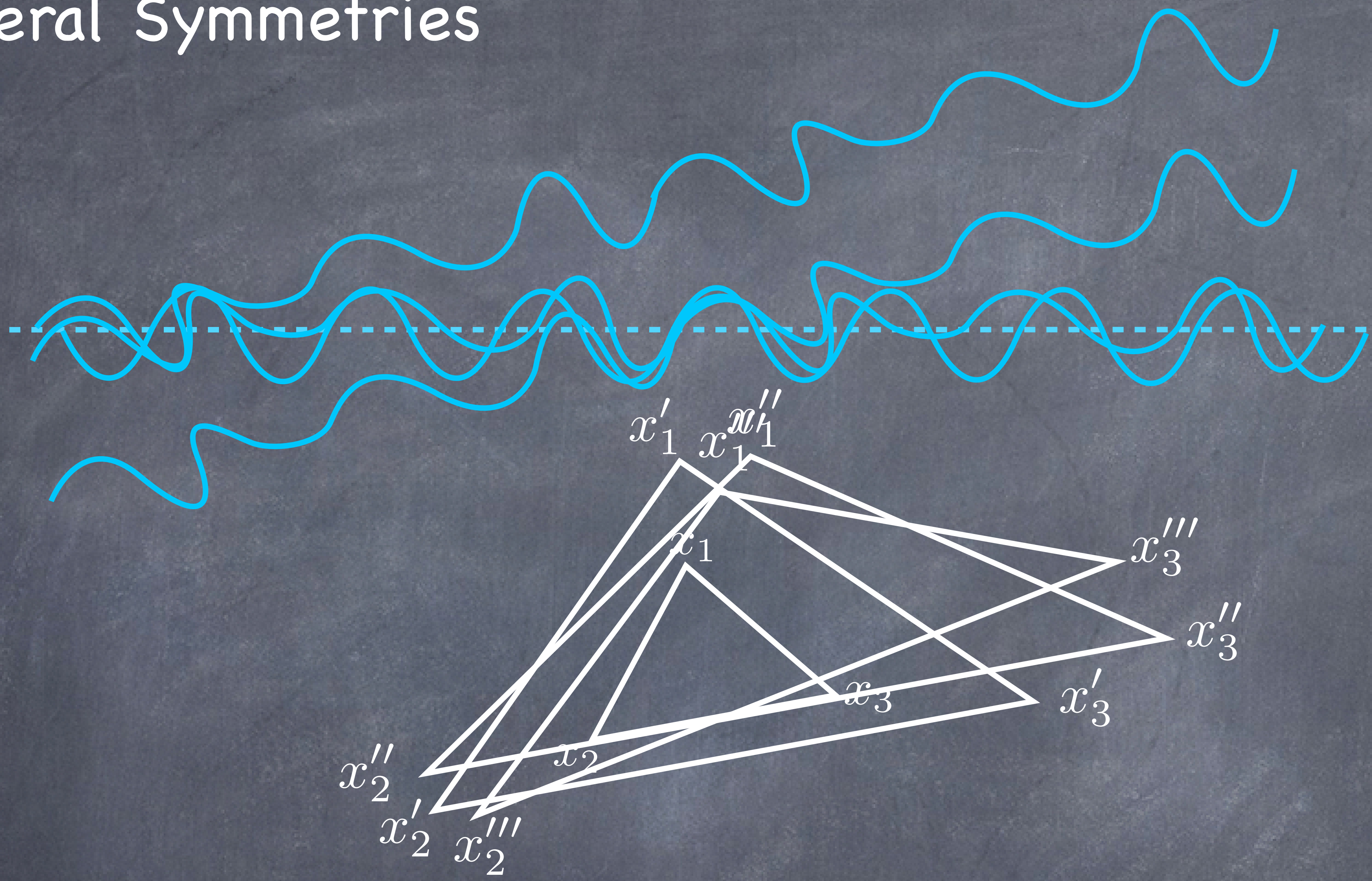
Single-field inflation constrained by infinite number of symmetries, corresponding to an infinite number of consistency relations:

$$\lim_{\vec{q}\to 0}\frac{\partial^n}{\partial q^n}\left(\frac{\langle\zeta_{\vec{q}}\mathcal{O}_{\vec{k}_1,\ldots,\vec{k}_N}\rangle}{P_\zeta(q)}+\frac{\langle\gamma_{\vec{q}}\mathcal{O}_{\vec{k}_1,\ldots,\vec{k}_N}\rangle}{P_\gamma(q)}\right)\sim\frac{\partial^n}{\partial k^n}\langle\mathcal{O}_{\vec{k}_1,\ldots,\vec{k}_N}\rangle$$

- q^0 and q behavior completely fixed (KNOWN)
- q^n, $n\geq 2$, behavior partially fixed (NEW)
- 3 identities for $n=0$; 7 identities for $n=1$
- Exactly 6 identities for all $n\geq 2$
- These are physical statements (i.e., can be violated)
- Hold on any spatially-flat FRW background (no slow-roll)

$\Longrightarrow$ Complete checklist for testing single-field mechanisms

General Symmetries



$$h_{ij} = \bar{\delta}_{ij} + \partial_k \bar{h}_{ij} x^k + \frac{1}{2} \partial_k \partial_\ell \bar{h}_{ij} x^k x^\ell + \dots$$

6 diffs at $\mathcal{O}(x^2)$

Master Consistency Relation

Berezhiani and Khoury, 1309.4461

(See also: Pimentel, 1309.1793)

Since symmetries of interest are subset spatial diffeomorphism, consistency relations must be consequence of gauge symmetry (Slavnov-Taylor identity).

E&M warm-up

Berezhiani and Khoury, 1309.4461

$$Z[J,\eta] = \int \mathcal{D}A_\mu \mathcal{D}\psi e^{iS_{\rm QED} - \frac{i}{2\xi}\int(\partial^\mu A_\mu)^2 + i\int(J^\mu A_\mu + \eta\psi)}$$

Field redefinition: $A_\mu \to A_\mu + \partial_\mu \Lambda\,; \qquad \psi \to \psi - i\Lambda\psi$

Z must be invariant: $\left[\frac{i\Box}{\xi}\partial^\mu \frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu + \eta\frac{\delta}{\delta\eta}\right] Z[J,\eta] = 0$

$\delta Z = 0$

Legendre transform ($J^\mu = -\frac{\delta\Gamma}{\delta A_\mu}$ etc.) :

$$-\frac{\Box}{\xi}\partial^\mu A_\mu + \partial_\mu \frac{\delta\Gamma}{\delta A_\mu} + i\psi\frac{\delta\Gamma}{\delta\psi} = 0$$

Can differentiate a number of times, e.g. $\Gamma_\mu^{A\bar\psi\psi} = \frac{\delta^3\Gamma}{\delta A^\mu \delta^2\psi}$,

$$q^\mu \Gamma_\mu^{A\bar\psi\psi}(q,p,-p-q) = \Gamma^\psi(p+q) - \Gamma^\psi(p)$$

(Ward-Takahashi)

$$q^\mu \Gamma_\mu^{A\bar{\psi}\psi}(q,p,-p-q) = \Gamma^\psi(p+q) - \Gamma^\psi(p)$$

General solution is power series:

$$\Gamma_\mu^{A\bar{\psi}\psi}(q,p,-p-q) = \sum_{n=0}^{\infty} q^{\alpha_1} \dots q^{\alpha_n} \frac{\partial^n \Gamma^\psi(p)}{\partial p^\mu \partial p^{\alpha_1} \dots \partial p^{\alpha_n}} + C_\mu$$

physical piece

$$q^\mu C_\mu = 0$$

If C_μ is analytic in q_μ (locality), then it drops out at $\mathcal{O}(q^0)$:

$$\Gamma_\mu^{A\bar{\psi}\psi}(0,p,-p) = \frac{\partial \Gamma_\psi(p)}{\partial p^\mu}$$

(QED analogue of Maldacena)

It can contribute at $\mathcal{O}(q^1)$, e.g. $C^\mu = q_\nu[\gamma^\nu, \gamma^\mu]$:

$$F_{\mu\nu}\bar{\psi}\gamma^\mu\gamma^\nu\psi$$

$\therefore$ C_μ encodes physical info about non-minimal couplings

Cosmological Slavnov-Taylor Identity

Berezhiani & Khoury, 1309.4461
Collins, Holman & Vardanyan, 1405.0017

Following similar steps,

$$2\partial_j\left(\frac{1}{6}\delta_{ij}\frac{\delta\Gamma}{\delta\zeta}+\frac{\delta\Gamma}{\delta\gamma_{ij}}\right)=\partial_i\zeta\frac{\delta\Gamma}{\delta\zeta}+\mathrm{G.F.}$$

Can vary this a number of times wrt the fields,
e.g. vary twice wrt ζ,

$$q^j\left(\frac{1}{3}\delta_{ij}\Gamma^{\zeta\zeta\zeta}+2\Gamma_{ij}^{\gamma\zeta\zeta}\right)=q_i\Gamma_\zeta(p)-p_i\Big(\Gamma_\zeta(|\vec{q}+\vec{p}|)-\Gamma_\zeta(p)\Big)$$

(Exact in q)

Analogue of W-T identity in E&M

General schematic solution:

$$\frac{1}{3}\delta_{ij}\Gamma^{\zeta\zeta\zeta}+2\Gamma_{ij}^{\gamma\zeta\zeta}=\sum_{n=0}^{\infty}q^n\frac{\partial^n}{\partial p^n}P_\zeta(p)+A_{ij}(\vec{p},\vec{q})$$

physical piece $q^j A_{ij}(\vec{p},\vec{q})=0$

Whether or not consistency relation holds hinges on model-dependent piece A_{ij} . Most general form:

$$A_{ij}(\vec{p},\vec{q}) = \epsilon_{ikm}\epsilon_{j\ell n}q^k q^\ell \left(a(\vec{p},\vec{q})\delta^{mn} + b(\vec{p},\vec{q})p^m p^n \right)$$

arbitrary scalar functions

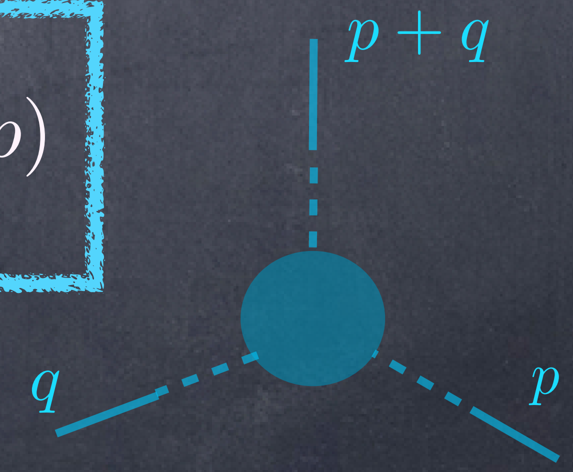
Key assumption: Suppose a and b are analytic in q, such that

$$A_{ij} = \mathcal{O}(q^2)$$

(Locality condition)

Then Maldacena's relation holds. Moreover, at each order in q can project out A_{ij} :

$$\lim_{\vec{q}\to 0} \frac{\partial^n}{\partial q^n}\left(\frac{\langle \zeta_{\vec{q}}\zeta_{\vec{p}}\zeta_{-\vec{q}-\vec{p}}\rangle}{P_\zeta(q)} + \frac{\langle \gamma_{\vec{q}}\zeta_{\vec{p}}\zeta_{-\vec{q}-\vec{p}}\rangle}{P_\gamma(q)} \right) \sim -\frac{\partial^n}{\partial p^n}P_\zeta(p)$$



General consistency relations

Physical Interpretation

Recall
$$\frac{1}{3}\delta_{ij}\Gamma^{\zeta\zeta\zeta} + 2\Gamma^{\gamma\zeta\zeta}_{ij} = \sum_{n=0}^{\infty} q^n \frac{\partial^n}{\partial p^n} P_\zeta(p) + A_{ij}(\vec{p},\vec{q})$$

cubic vertices

$\Longrightarrow$ $A_{ij} = \mathcal{O}(q^2)$ is a locality requirement on the action.

- Naively this seems trivially satisfied, since GR + ϕ is a local theory

- But it's not: we have already integrated out N and N^i, hence the action $S = S[\zeta, \gamma_{ij}]$ is non-local.

$$N_i \supset -a^2 \frac{\dot{H}}{H^2} \frac{q_i}{q^2} \dot{\zeta}$$

- For adiabatic modes, $\dot{\zeta} \propto q^2$ $\Longrightarrow$ Locality

Consistency Relations and the Initial State

Berezhiani & JK, 1406.2689
Collins, Holman & Vardanyan, 1405.0017

- Possible to violate consistency relation with non-Bunch-Davies initial states, e.g. Agarwal, Holman, Tolley & Lin, 1212.1172.
- Where does that come into the background-wave argument?
- Goldberger et al's derivation seems to apply to any gauge inv state.

Lam's paradox:

- Take some multi-field scenario, which generates significant $f_{\rm NL}^{\rm local}$.
- For the consistency relation derivation, choose "initial time" well after inflation.
- Subsequently, have single fluid, $\zeta \simeq {\rm const.}$ etc.

$\Longrightarrow$ Consistency relations should hold???

Consistency Relations and the Initial State

Berezhiani & JK, 1406.2689

In the in-in formalism,

$$Z[J^+,J^-]=\int\mathcal{D}\Phi^+\,\mathcal{D}\Phi^-\,\exp\left[i\Big(S[\Phi^+,J^+]-S[\Phi^-,J^-]\Big)\right]\rho(\Phi^+,\Phi^-;t_0)$$

density mtx

Focusing on pure states, $\rho(\Phi^a;t_0)\sim\exp\left[i\Big(\mathcal{S}\left[\Phi^+;t_0\right]-\mathcal{S}\left[\Phi^-;t_0\right]\Big)\right]$

$$\Longrightarrow\quad \Gamma[\Phi^+,\Phi^-]=S[\Phi^+]+\mathcal{S}[\Phi^+;t_0]-S[\Phi^-]-\mathcal{S}[\Phi^-;t_0]$$

Slavnov-Taylor applies separately to S and $\mathcal{S}$, e.g.

$$\sum_{\pm}\left[2\partial_j\frac{\delta\mathcal{S}}{\delta h^{\pm}_{jk}}-\partial_k h^{\pm}_{ij}\frac{\delta\mathcal{S}}{\delta h^{\pm}_{ij}}+2\partial_j\left(h^{\pm}_{ik}\frac{\delta\mathcal{S}}{\delta h^{\pm}_{jk}}\right)\right]=0$$

Violations of consistency relation due to initial state trace back to non-localities in $\mathcal{S}$.

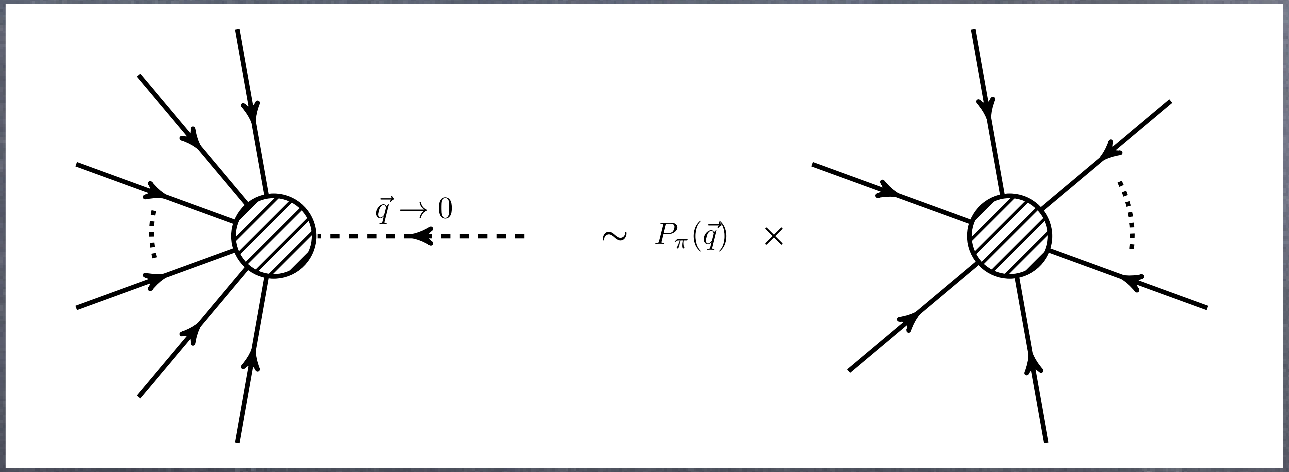
Consistency Relations in the Conformal Alternative

Creminelli, Joyce, JK and Simonovic, 1212.3329

- Conformal mechanism:
  - Quasi-static universe
  - Scale invariance from conformal invariance

$$so(4,2) \to so(4,1)$$

- Soft pion thms (Ward identities) from the 5 broken symmetries



$$\lim_{\vec{q}\to 0} \frac{1}{P_\pi(q)} \langle \pi(\vec{q}) \mathcal{O}(\vec{k}_a) \rangle = -\left(1 + \frac{1}{N} \sum_a \vec{q} \cdot \frac{\partial}{\partial \vec{k}_a} + \frac{q^2}{6N} \sum_a \frac{\partial^2}{\partial k_a^2} \right) t \frac{\partial}{\partial t} \langle \mathcal{O}(\vec{k}_a) \rangle$$

Multiple Soft Limits

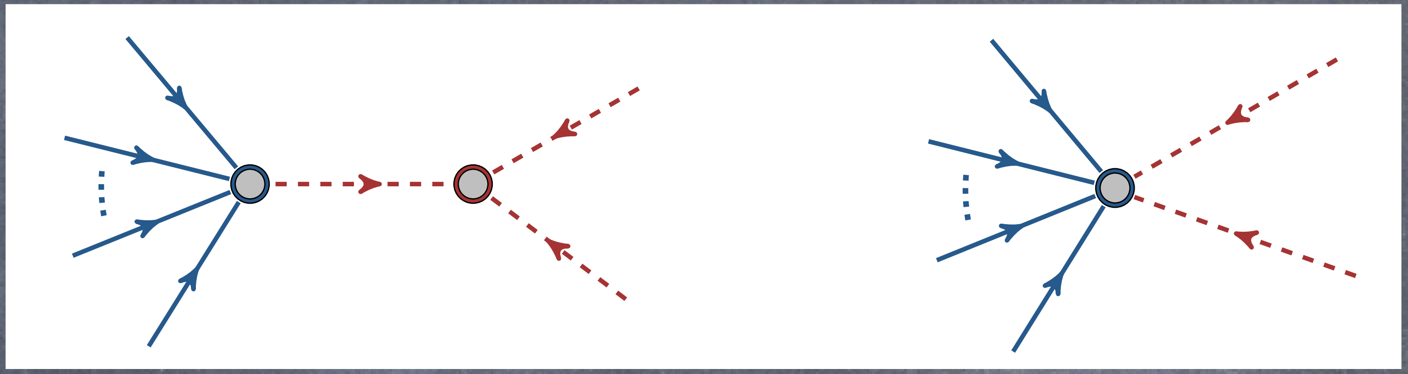
Senatore & Zaldarriaga, 1203.6884
Chen, Huang & Shiu, hep-th/0610235
Joyce, JK & Simonovic, to appear

Another probe of higher-q dependence.

e.g. Strong interactions:

$$\lim_{q_a,q_b\to 0}\langle \pi^a(q_a)\pi^b(q_b)\pi^{i_1}(k_1)\cdots\pi^{i_n}(k_n)\rangle = \frac{1}{2}\sum_j \frac{(q_a-q_b)\cdot k_j}{(q_a+q_b)\cdot k_j}\epsilon^{abc}\langle \pi^{i_1}(k_1)\cdots T_c\pi^{i_j}(k_j)\cdots\pi^{i_n}(k_n)\rangle$$

Double-soft result:



$$\lim_{\vec q_1,\vec q_2\to 0}\frac{\langle \zeta_{\vec q_1}\zeta_{\vec q_2}\zeta_{\vec k_1}\cdots\zeta_{\vec k_N}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)} = \frac{\langle \zeta_{\vec q_1}\zeta_{\vec q_2}\zeta_{-\vec q}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}\left(\delta_{\mathcal D}+\frac{1}{2}\vec q_1\cdot\delta_{\vec{\mathcal K}}\right)\langle\zeta_{\vec k_1}\cdots\zeta_{\vec k_N}\rangle'$$
$$+\left(\delta^2_{\mathcal D}+\frac{1}{2}\vec q_1\cdot\delta_{\vec{\mathcal K}}\delta_{\mathcal D}+\frac{1}{4}q_1^i q_2^j\delta_{\mathcal K^i}\delta_{\mathcal K^j}\right)\langle\zeta_{\vec k_1}\cdots\zeta_{\vec k_N}\rangle'$$
$$+\lim_{\vec q\to 0}\left[\frac{1}{2}\left(\vec q^{\,2}\nabla_q^2-2q_iq_j\nabla_q^i\nabla_q^j\right)\langle\zeta_{\vec q}\zeta_{\vec k_1}\cdots\zeta_{\vec k_N}\rangle'+\frac{\langle \zeta_{\vec q_1}\zeta_{\vec q_2}\zeta_{-\vec q}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}q_iq_j\nabla_q^i\nabla_q^j\frac{\langle\zeta_{\vec q}\zeta_{\vec k_1}\cdots\zeta_{\vec k_N}\rangle'}{P_\zeta(q)}\right]$$

$\delta_{\mathcal D}\equiv$ dilation $\qquad \delta_{\mathcal K}\equiv$ SCT

Large Scale Structure

Kehagias & Riotto, 1302.0130; Peloso & Pietroni, 1302.0223; Creminelli, Norena & Simonovic, 1309.3557
Horn, Hui & Xiao, 1406.0842

The inflationary consistency relations translate at late times to consistency relations for the LSS.

When short modes are deep inside Hubble, the relevant symmetry is

$$\eta \to \eta \,, \qquad \vec{x} \to \vec{x} + \frac{1}{6}\eta^2 \vec{\nabla}\Phi_{\rm L}$$

homogeneous acc'n

$\Longrightarrow$ Equiv. Principle!

$$\lim_{\vec{q}\to 0} \langle \delta_{\vec{q}}(\eta)\, \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle = -P_\delta(q,\eta) \sum_a \frac{D(\eta_a)}{D(\eta)} \frac{\vec{q}\cdot\vec{k}_a}{q^2} \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle$$

- Only assumes $\delta_{\vec{q}} \ll 1$
- The short modes can be highly non-linear, including bias issues, messy astrophysics etc.

Conclusions

- Single-field inflation constrained by infinitely-many relations (indep. of slow-roll, c_s, ϕ fundamental or not)

$$\lim_{\vec{q}\to 0}\frac{\partial^n}{\partial q^n}\left(\frac{\langle\zeta_{\vec{q}}\mathcal{O}_{\vec{k}_1,\ldots,\vec{k}_N}\rangle}{P_\zeta(q)}+\frac{\langle\gamma_{\vec{q}}\mathcal{O}_{\vec{k}_1,\ldots,\vec{k}_N}\rangle}{P_\gamma(q)}\right)\sim\frac{\partial^n}{\partial k^n}\langle\mathcal{O}_{\vec{k}_1,\ldots,\vec{k}_N}\rangle\,.$$

- All follow from Slavnov-Taylor identity for spatial diffs

- Open questions:

    - Other symmetries?

    - Ward identities for open inflation?

    - Impact of modified initial state on LSS consistency relations?

    - Multiple soft identities with tensors?