Nongaussianity from Nonlocality

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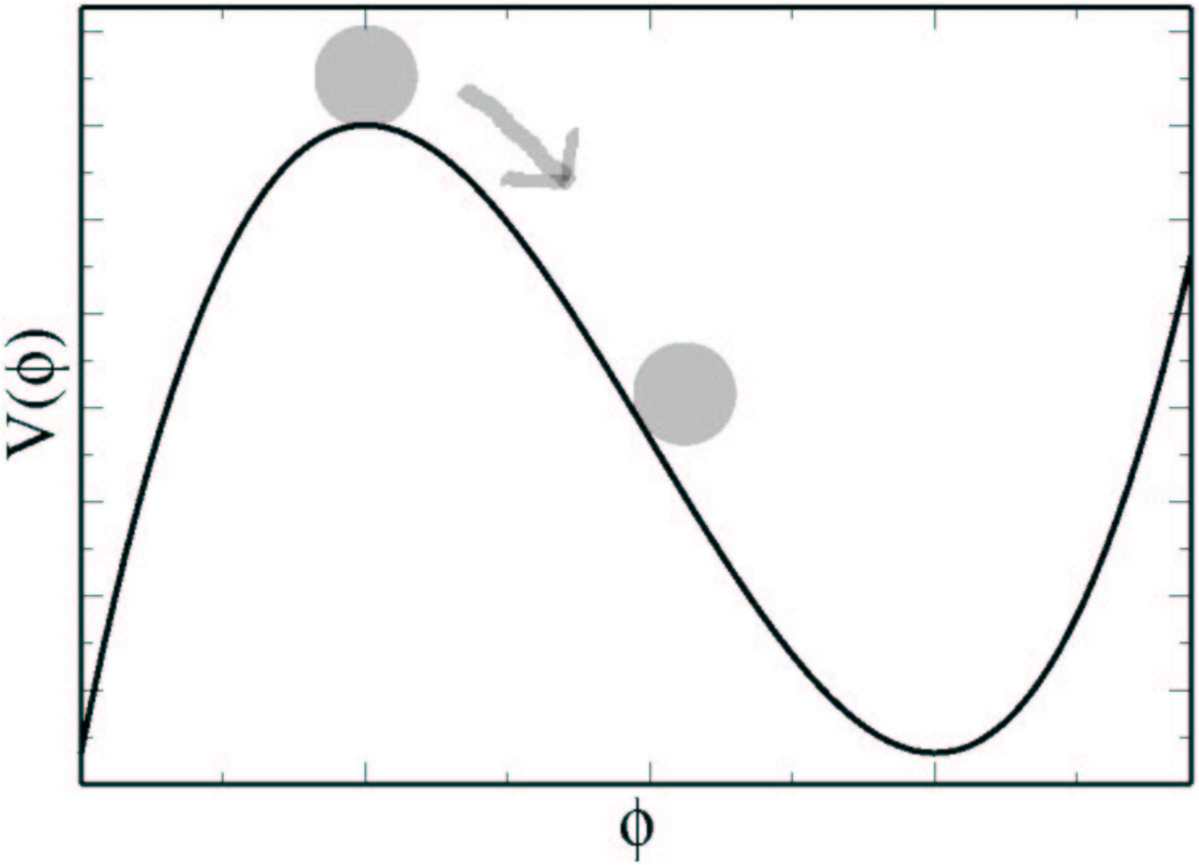
Based on:

1. ”p-adic inflation", NB, T. Biswas and J. Cline, JHEP 0704, 056; arXiv:hep-th/0612230.
2. ”Large Nongaussianity from Nonlocal Inflation", NB and J. Cline; arXiv:0704.3426.
3. NB & N. Kamran; work in progress.

Outline

1. Nonlocal Hill-Top Models
2. Ghosts and Ostrogradski Instability
3. Nonlocal Hill-Top Inflation
4. Nongaussianity from Nonlocality

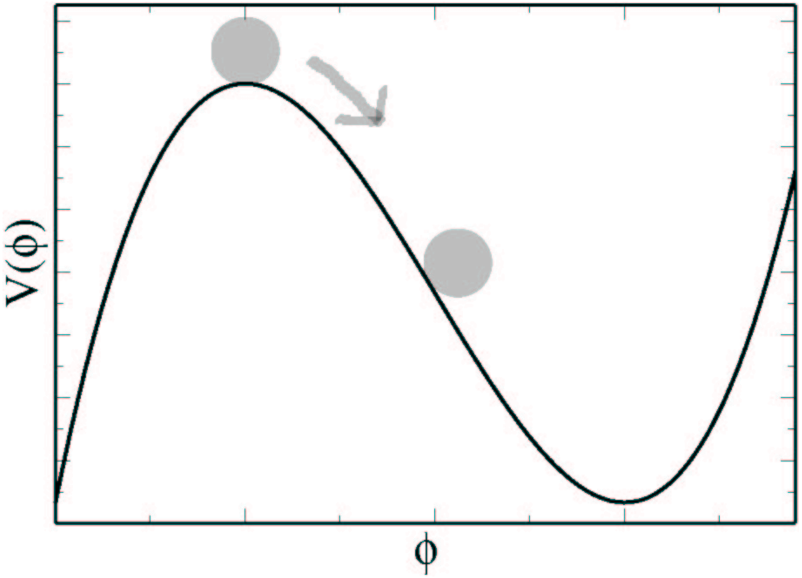
Part 1: Nonlocal Hill-Top Models



Nonlocal Hill-Top Inflation

- ⋆ Consider general nonlocal theories of the form:

$$\begin{aligned}\mathcal{L} &= \gamma^4\left[\frac{1}{2}\phi F\left(\frac{\Box}{m_s^2}\right)\phi - U(\phi)\right]\\ F\left(\frac{\Box}{m_s^2}\right) &= \sum_{n=1}^{\infty} c_n\left(\frac{\Box}{m_s^2}\right)^n\\ U(\phi) &= U_0 - \frac{\mu^2}{2}\phi^2 + \frac{g}{3}\phi^3 + \cdots\end{aligned}$$



- ⋆ Seek inflationary solution rolling away from $\phi = 0$.
- ⋆ Can derive such actions from string theory.

Example: p-adic String Theory

- Toy model of the bosonic open string tachyon.[a]
- World-sheet coordinates of the string are restricted to the field of p-adic numbers.
- All amplitudes of the lowest state can be computed exactly and one can determine a simple field-theoretic Lagrangian which reproduces them:

$$\mathcal{L} = \frac{m_s^4\, p^2}{g_s^2\,(p-1)} \left[-\frac{1}{2}\psi p^{-\frac{\Box}{2m_s^2}}\psi + \frac{1}{p+1}\psi^{p+1} \right]$$

- Nonlocal field theory which is sensible in the far UV.
- Derived for p a prime number but the theory can be sensibly continued to other values.

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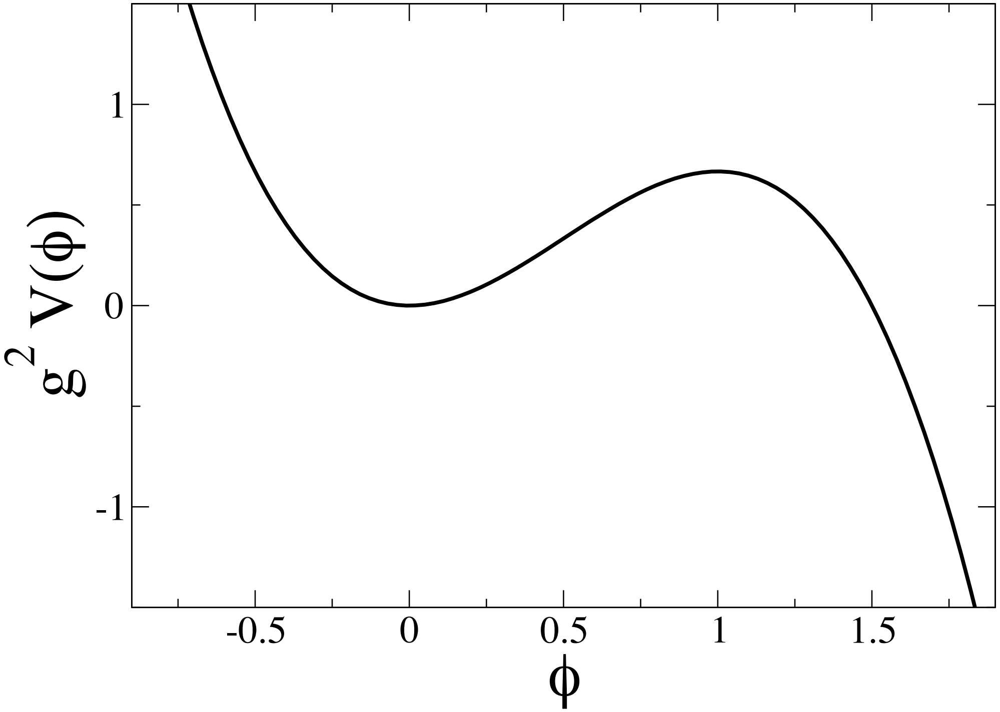
[a] Brekke, Freund, Olson & Witten (1987).

p-adic Potential

$$V(\psi) = \frac{m_s^4}{g_p^2}\left[\frac{1}{2}\psi^2 - \frac{1}{p+1}\psi^{p+1}\right]$$

- ⋆ $\psi = 1$ (and $\psi = -1$ for odd p) is the unstable maximum: D25 brane
- ⋆ $\psi = 0$ is the true vacuum: no brane, open strings
- ⋆ $V(\psi)$ is unbounded from below.



- ⋆ Instability to tunnel from $V = 0$ to $V = -\infty$ corresponds to the closed string tachyon

Limit of Local Field Theory

- ⋆ The field equation for the p-adic scalar is:

$$p^{-\frac{\Box}{2m_s^2}}\psi = \psi^p$$

- ⋆ In the limit $p \to 1$ this equation becomes local:[a]

$$\Box\psi = 2m_s^2\psi\ln\psi$$

- ⋆ For $p \gg 1$ the nonlocal structure plays an important role in the dynamics. (Will require $p \gg 1$ to fit CMB observations.)
- ⋆ Infinitely many derivatives is fundamentally different from N derivatives with $N \gg 1$.[b]

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[a] Gerasimov & Shatashvili (2000).

[b] Moeller & Zwiebach (2002).

The Initial Value Problem

- Consider a homogeneous configuration:

$$p^{\frac{1}{2} m_s^{-2} \partial_t^2} \psi(t) = \psi^p(t)$$

- For an equation with N derivatives we can choose N initial conditions ($\psi(0)$ and its $N-1$ derivatives).
- Naively might expect that for $N \to \infty$ we can choose $\psi(0)$ and all its derivatives.
- But a contradiction arises: the assumption of analyticity implies that the complete $\psi(t)$ could be reconstructed by a Taylor expansion!

$$\psi(t) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!} t^n$$

- EOM places infinitely many constraints on allowed IC so that space of allowed IC is finite![a]

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[a] NB & Kamran, work in progress.

Examples from String Theory

- ⋆ Nonlocal hill-top models:
  - – p-adic string theory.
  - – String field theory tachyon.

$$\mathcal{L} = \frac{m_s^4}{g_s^2}\left[\frac{1}{2}\psi\left(1+\lambda^2\frac{\Box}{m_s^2}\right)e^{-\frac{\Box}{4m_s^2}}\psi - \frac{1}{4}\psi^4\right]$$

  - – Strings quantized on a random lattice.[a] (The p-adic string provides a discretization of the bosonic string worldsheet.[b])
  - – $\ldots$
- ⋆ Focus on p-adic inflation but our analysis applies more generally.

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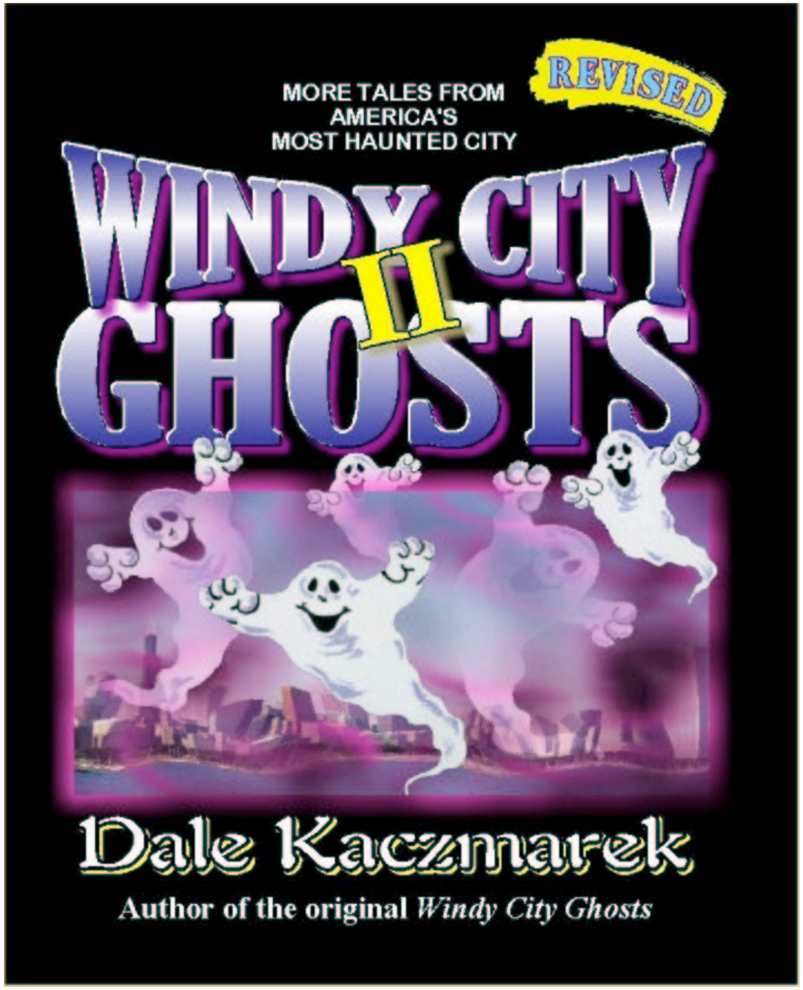
[a] Douglas & Shenker (1990), Gross & Migdal (1990), Brezin & Kazakov (1990), Biswas, Grisaru & Siegel (2005).

[b] Ghoshal (2006).

Part 2: Ghosts and Ostrogradski Instability



Ghosts

- Higher derivative theories often considered sick.
- Finite extra derivative theories generically contain ghost excitations which either
  - violate unitarity, or,
  - carry negative kinetic energy and lead to vacuum instability.

$$\mathcal{L}_{\text{ghost}} = +\frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Ghosts

- ⋆ Consider some higher derivative equation:

$$\left(\alpha\Box^2 - \Box + m^2\right)\phi = 0$$

- ⋆ The propagator is:

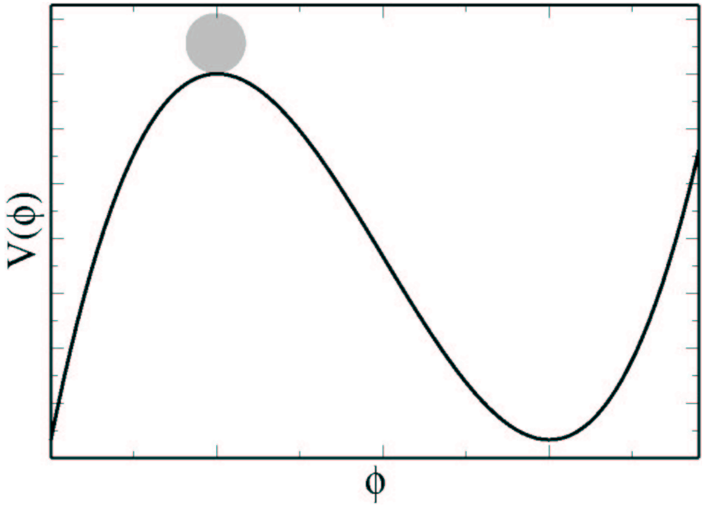
$$\begin{aligned} G(k^2) &\sim \frac{1}{\alpha k^4 + k^2 + m^2} \\ &\propto \frac{1}{k^2 + m_+^2} - \frac{1}{k^2 + m_-^2} \end{aligned}$$

- ⋆ Two poles corresponding to two physical states with mass $m_\pm$.
- ⋆ Relative minus sign ⇒ one state with wrong-sign kinetic term!
- ⋆ What about infinitely many derivatives? p-adic string?

No Ghosts in the False Vacuum

- ⋆ p-adic theory linearized about $\psi = 1$:

$$\left(p^{-\frac{1}{2}m_s^{-2}\Box} - p\right)\psi = 0$$



- ⋆ The propagator is:

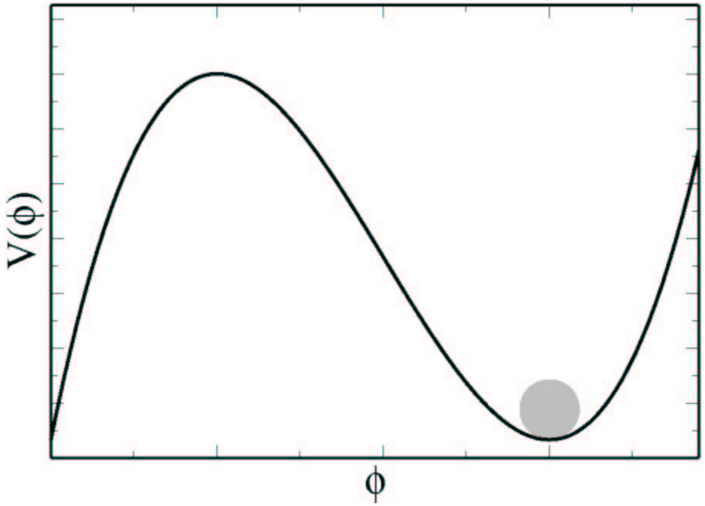
$$G(k^2) \sim \frac{1}{p^{k^2/(2m_s^2)} - p}$$

- ⋆ One pole with mass-squared $-2m_s^2$: the tachyon!
- ⋆ Don't introduce ghosts because the propagator is modified nonperturbatively.

No Open Strings in the True Vacuum

- ⋆ p-adic theory linearized about $\psi = 0$:

$$p^{-\frac{1}{2}m_s^{-2}\Box}\psi = 0$$



- ⋆ The propagator is:

$$G(k^2) \sim \frac{1}{p^{k^2/(2m_s^2)}}$$

- ⋆ No poles ⇒ No physical states in the true vacuum!
- ⋆ p-adic version of the statement that open strings require a brane to end on.

Ostrogradski Instability

- Classical manifestation of having ghosts in the theory.
- For a theory with N derivatives the Hamiltonian depends on N canonical coordinates (just like p, q in $N = 2$ theories).
- Theorem: the Hamiltonian always depends linearly on $N - 1$ of the conjugate momenta $\Rightarrow \mathcal{H}$ is unbounded from below.
- Higher derivative theories are generically unstable.
- This is why Newton's laws involve no more than two time derivatives of the fundamental dynamical variables:

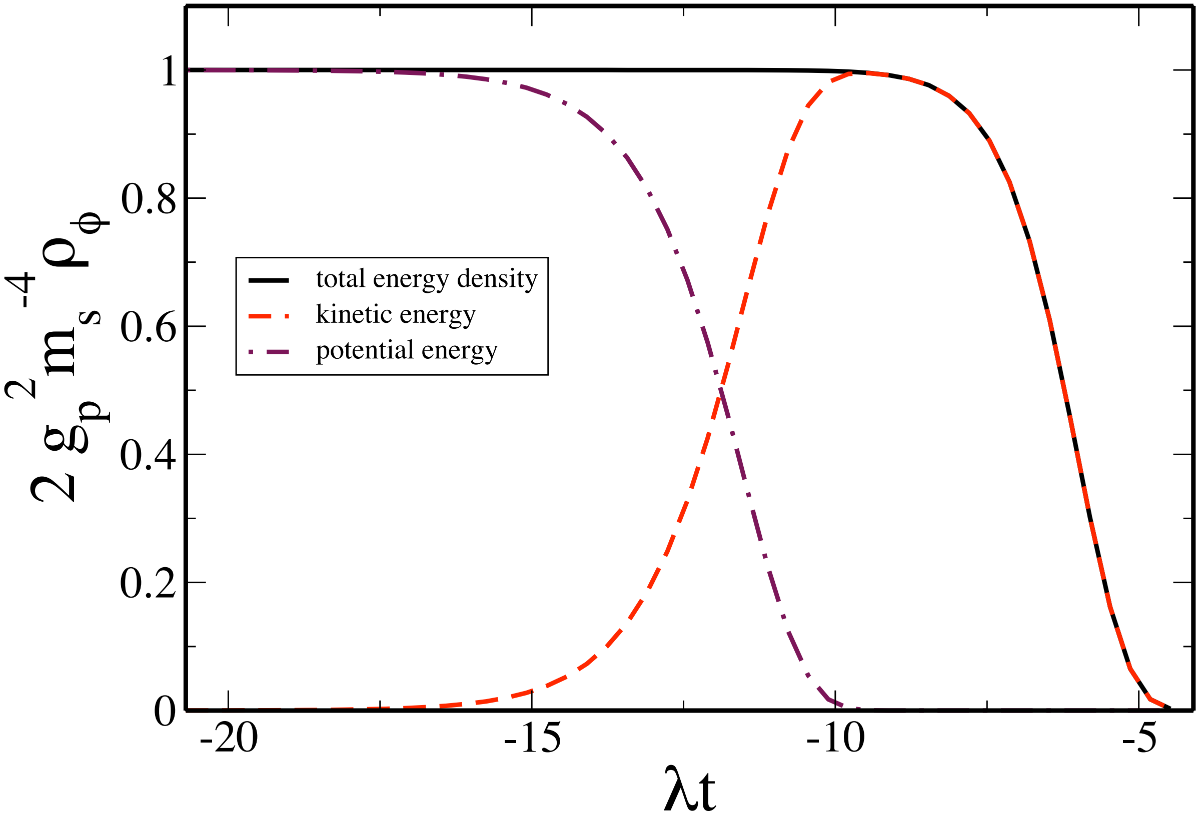
$$F = ma + \cancel{\kappa \dot{a}}$$

Ostrogradski Instability

- ⋆ The Ostrogradski construction does not apply to the p-adic theory.
- ⋆ For p-adic strings the conjugate momenta are not all independent!
  - This is for the same reason that the initial value problem does not admit infinitely many Cauchy data.
- ⋆ Models derived from string theory are free of the problems which typically plague higher derivative theories.

Note: the fact that $V(\psi)$ is unbounded from below is related to the closed string tachyon and has nothing to do with the Ostrogradski instability.

Part 3: Nonlocal Hill-Top Inflation



Free Theory

Near the top of the potential ($\phi = 0$) have:

$$\mathcal{L} = \gamma^4 \left[\frac{1}{2} \phi F\left(\frac{\Box}{m_s^2}\right)\phi - \left(U_0 - \frac{\mu^2}{2}\phi^2 + \underbrace{\cdots}_{\text{interactions}}\right)\right]$$

- ⋆ Equation of motion:

$$F\left(\frac{\Box}{m_s^2}\right)\phi = -\mu^2\phi$$

- ⋆ Can obtain solution by taking:

$$\Box\phi = -\omega^2\phi \quad \text{if} \quad F\left(-\omega^2/m_s^2\right) = -\mu^2$$

- ⋆ Effective mass: $\omega^2 = -m_s^2 F^{-1}(-\mu^2)$.

Evading Ghosts

- Equation of motion:

$$F\left(\frac{\Box}{m_s^2}\right)\phi = -\mu^2\phi \quad \Leftrightarrow \quad \Box\phi = -\omega^2\phi$$

- In general $F(-\omega^2/m_s^2) = -\mu^2$ could admit multiple solutions for ω^2.
- Propagator:

$$G(k) \sim \frac{1}{F(-k^2/m_s^2) - \mu^2}$$

- Restriction to single-valued $\omega^2 = -m_s^2 F^{-1}(-\mu^2)$ ensures absence of ghosts, Ostrogradski instabilities.

Inflationary Dynamics

- ⋆ Near $\phi = 0$ the dynamics are governed by:

$$\begin{aligned} 3H^2 &\cong \frac{\gamma^4 U_0}{M_p^2} \\ \Box\phi &\cong -\omega^2\phi \end{aligned}$$

- ⋆ Slow roll condition in terms of effective mass:

$$|\eta| = \frac{|\omega^2|}{3H^2} = \frac{m_s^2}{3H^2}|F^{-1}(-\mu^2)| \ll 1$$

- ⋆ With suitable choice of $F(z)$ can achieve slow roll inflaton even when, naively, the potential is extremely steep!

Stretching the Inflaton Potential

$$\mathcal{L} \;=\; \gamma^4 \left[\frac{1}{2}\phi\, F\left(\frac{\Box}{m_s^2}\right)\phi - U_0 + \frac{\mu^2}{2}\phi^2 + \cdots\right]$$

$$\Updownarrow \qquad \text{On shell, near } \phi = 0$$

$$\mathcal{L} \;=\; \frac{1}{2}\varphi\Box\varphi - \gamma^4 U_0 + \underbrace{\frac{\omega^2}{2}\varphi^2}_{=-m_s^2 F^{-1}(-\mu^2)} + \cdots$$

- ⋆ The canonical inflaton is: $\varphi = \frac{\gamma^2\mu}{\omega}\,\phi$.
- ⋆ Close to $\phi = 0$ the full nonlocal theory is equivalent to a dual, local theory with a stretched potential.[a]
- ⋆ Usual inflatonary calculation goes through for canonical inflaton, φ.

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[a] Lidsey (2007).

Example: p-adic Inflation

Explicit example in p-adic string theory:[a]

$$\begin{aligned} \mathcal{L} &= \frac{m_s^4}{g_p^2}\left[\frac{1}{2}\phi\left(1-p^{-\frac{\Box}{2m_s^2}}\right)\phi - U(\phi)\right] \\ U(\phi) &= \underbrace{\frac{p-1}{2(p+1)}}_{\equiv U_0} - \underbrace{\frac{p-1}{2}}_{\mu^2 \equiv p-1 \gg 1}\phi^2 + \cdots \end{aligned}$$

- ⋆ **Effective mass**: $\omega^2 = -2m_s^2$, insensitive to p.
- ⋆ Slow roll parameter:

$$|\eta| = \frac{|\omega^2|}{3H^2} = \frac{2m_s^2}{3H^2} \ll 1$$

- ⋆ With $N_e \cong 60$, $n_s \cong 0.95$ the COBE normalization gives $g_s/\sqrt{p} \sim 10^{-7}$ so for $g_s \sim 1$ expect $p \gg 1$.

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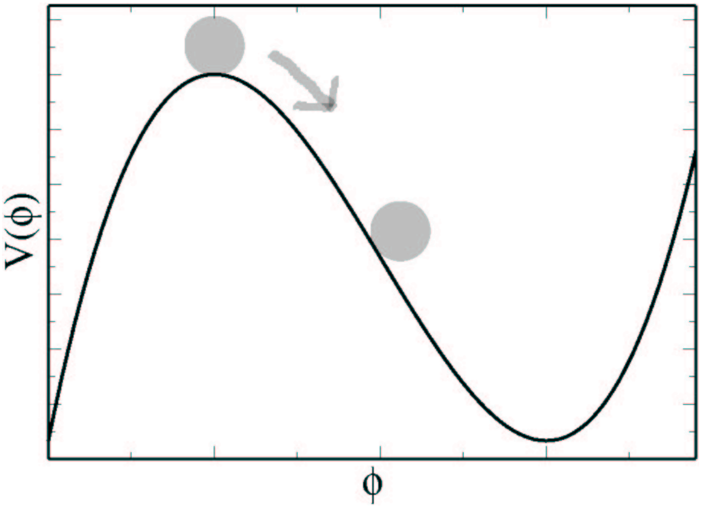
[a] NB, Biswas & Cline (2006).

Slow Roll with a Steep Potential

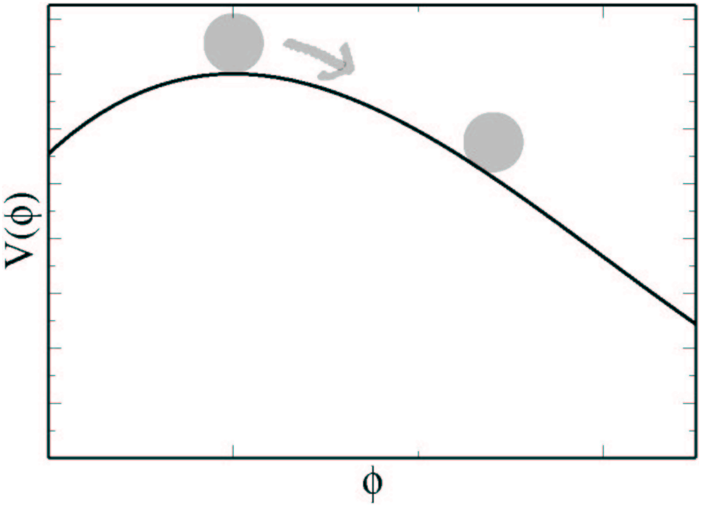
- ⋆ Might expect that during slow roll the higher powers $\Box^n$ are small. Not true for $p \gg 1$!



$$\mathcal{L} = \frac{1}{2}\chi\Box\chi - V(\chi) + \mathcal{O}(\Box^2) + \cdots$$

$$V(\chi) = \gamma^4 U_0 - \frac{m^2}{2}\chi^2 + \cdots$$

Naive potential inferred from derivative truncation is extremely steep, $|\eta_\chi| \sim 10^{11}$.



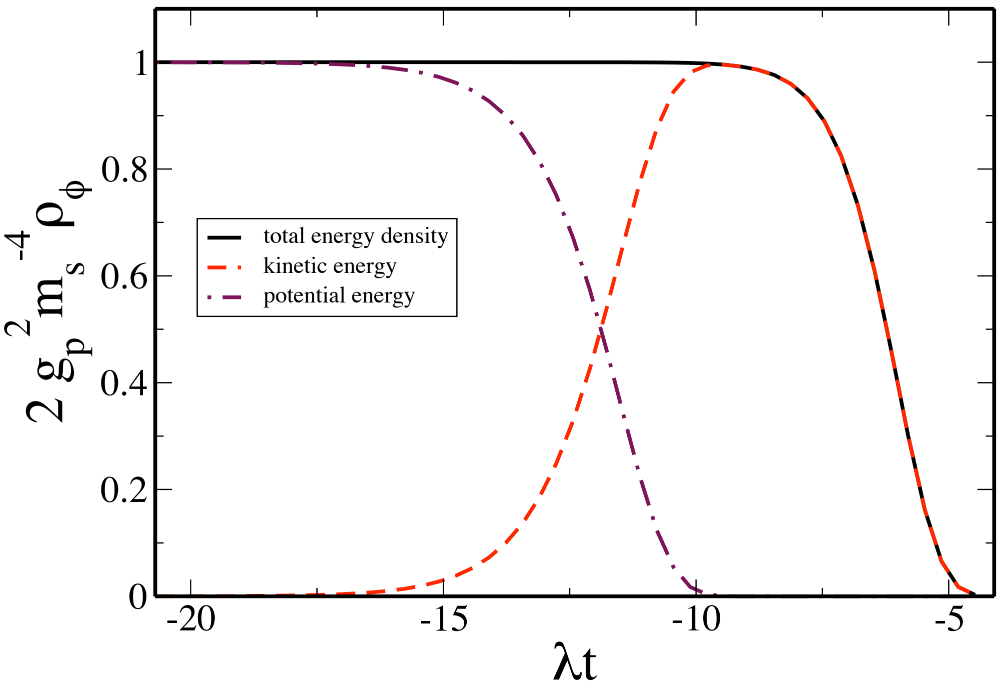
$$\mathcal{L}_{\mathrm{on-shell}} = \frac{1}{2}\varphi\Box\varphi - V(\varphi)$$

$$V(\varphi) = \gamma^4 U_0 - \frac{\omega^2}{2}\varphi^2 + \cdots$$

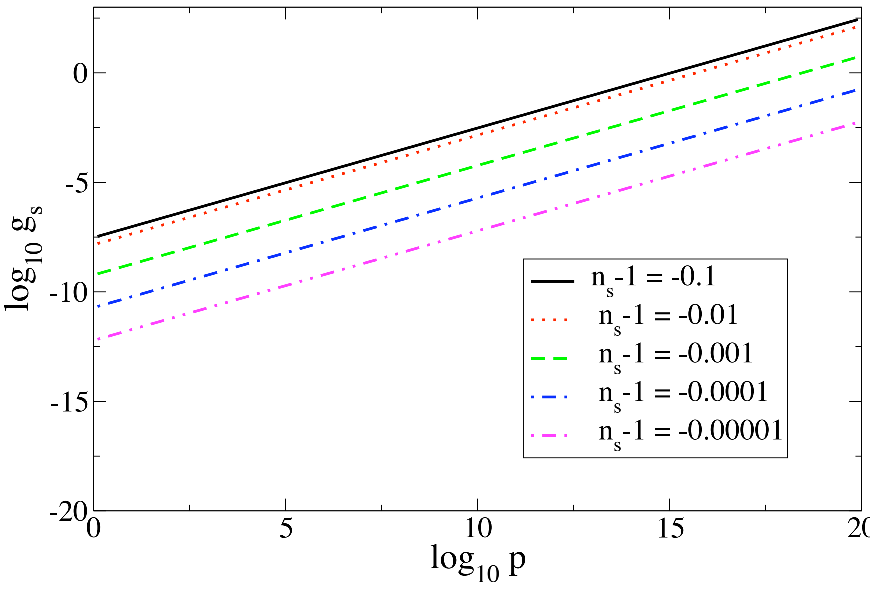
Effective potential in dual local theory is flattened, $|\eta_\varphi| \sim 0.05$.

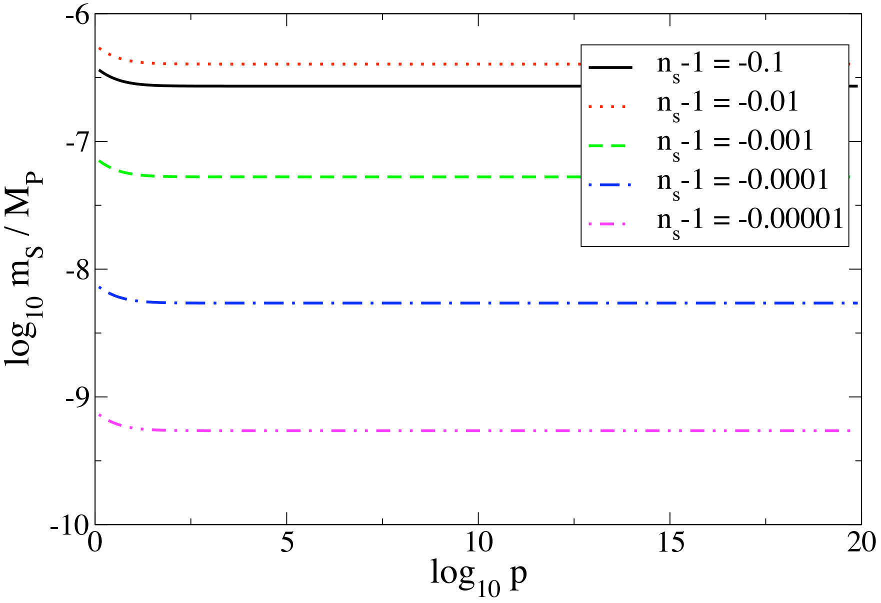
p-adic Inflation

- ⋆ Can also study dynamics in an approximation which doesn't rely on small ϕ.
- ⋆ Friction dominated approximation: $\Box \cong -3H\partial_t$.
- ⋆ p-adic EOM becomes algebraic: $\phi(t+\alpha) = \phi(t)^p \Rightarrow \phi(t) = \exp\left[\exp(-\lambda t)\right]$.



Predictions for the CMB



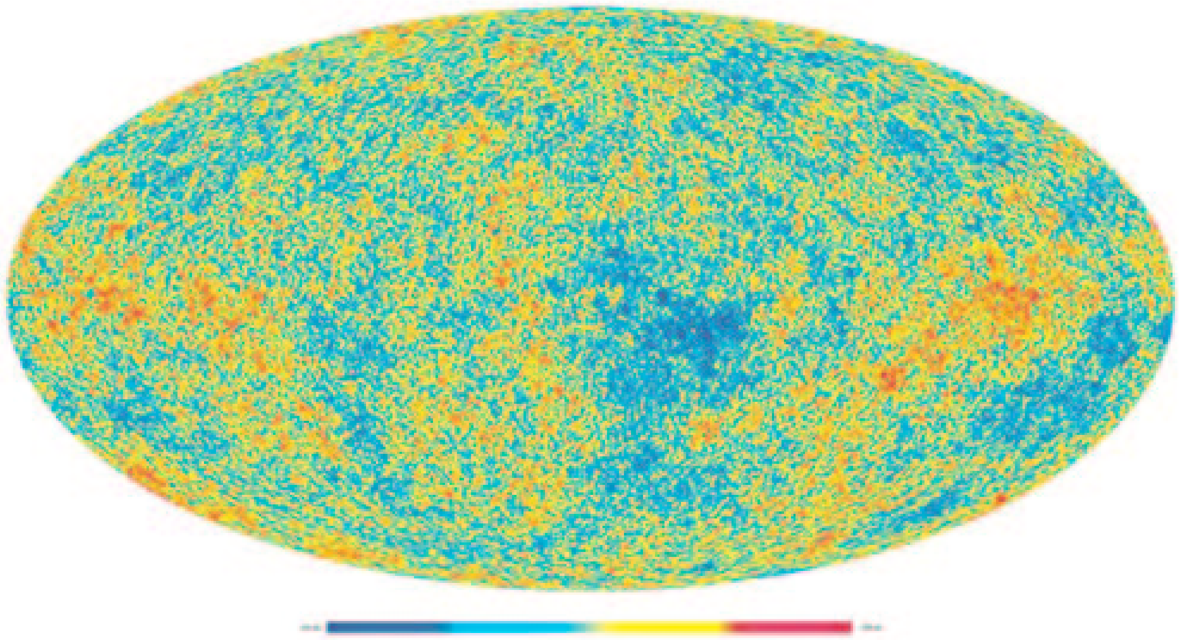


- ⋆ COBE normalization:

$$\frac{g_s}{\sqrt{p}} \cong 10^{-7}$$

- ⋆ String scale bounded as $m_s \lesssim 10^{-6} M_p$.
- ⋆ Tensor modes undetectably small: $r \lesssim 0.006$.

Part 5: Large Nongaussianity from Nonlocality



Including Interactions

Include the cubic term in the action:

$$\begin{aligned} \mathcal{L} &= \gamma^4 \left[\phi F\left(\frac{\Box}{m_s^2}\right)\phi - U(\phi)\right] \\ U(\phi) &= U_0 - \frac{\mu^2}{2}\phi^2 + \frac{g}{3}\phi^3 + \cdots \end{aligned}$$

- ⋆ For $g \neq 0$ the correspondence between local and nonlocal theories breaks down.
- ⋆ Expect $\langle \phi^3 \rangle \propto g$ so for large g the nongaussianity could be large.
- ⋆ In conventional models $g \gg 1$ would spoil inflaton but this need not be true in nonlocal theories!
- ⋆ In p-adic inflation:

$$|g| \sim p^2 \gg 1 \quad \text{for} \quad g_s \sim 1$$

Including Interactions

Perturbed Lagragian, neglecting metric perturbations:

$$\begin{aligned}\mathcal{L}_{\text{free}} &= \frac{\omega^2}{2\mu^2}\,\delta\varphi\, F\left(\frac{\Box}{m_s^2}\right)\,\delta\varphi + \frac{\mu^2}{2}\,(\delta\varphi)^2 \\ \mathcal{L}_{\text{int}} &= \underbrace{c_H}_{\propto g}\,(\delta\varphi)^3\end{aligned}$$

- ⋆ Perturbed field equation:

$$F\left(\frac{\Box}{m_s^2}\right)\delta\varphi = -\mu^2\delta\varphi \quad \Rightarrow \quad \Box\delta\varphi = -\omega^2\delta\varphi$$

- ⋆ Assume curvature perturbation:

$$\zeta \sim -\frac{H}{\dot{\varphi}}\delta\varphi$$

- ⋆ Calculation of $\langle\zeta^3\rangle$ equivalent to computing three-point function for a light field in de Sitter.

The Nonlinearity Parameter

- ⋆ WMAP ansatz for bispectrum:

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (2\pi)^7 \left(-\frac{3}{10} f_{NL}\right) A_\zeta^4 \frac{k_1^3+k_2^3+k_3^3}{k_1^3 k_2^3 k_3^3}\delta(k_1+k_2+k_3)$$

where $A_\zeta^2 \cong 25\times 10^{-10}$.

- ⋆ Nonlinearity parameter:[a]

$$f_{NL} \cong -\frac{5N_e}{48\pi^3}\, g\, U_0^{1/2} e^{-\frac{N_e}{2}|n_s-1|} |n_s-1|^2 K(\mu^2)$$

with $K(\mu^2)$ a complicated function depending on details of the kinetic function $F(z)$.

Note: $f_{NL} \propto g$ so max nongaussianity comes from largest g for which calculation is reliable.

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[a] NB & Cline (2007).

The Perturbative Regime

- ⋆ Potential:

$$V(\varphi) = V_0 - \frac{\omega^2}{2}\varphi^2 + \underbrace{c_H\varphi^3}_{c_H \propto g}$$

- ⋆ Calculation is under perturbative control for $c_H^2 < \omega^2$ which bounds g from above:

$$g < \frac{3\mu^2}{|F^{-1}(-\mu^2)|}\frac{\gamma^2}{m_s^2}$$

- ⋆ $f_{NL} \propto g$ also bounded from above.
- ⋆ Can we obtain $|f_{NL}| \gg 1$ while keeping the calculation under control?

Large Nongaussianity

- For p-adic inflation:
  - With $N_e \cong 60$, $n_s \cong 0.95$ have:

$$f_{NL} \sim 120\, g_s$$

  - Require $g_s < 1$ for the p-adic action to be reliable so

$$f_{NL} \lesssim 120$$

  - $f_{NL} \gg 1$ seems natural since corresponds to $g_s = \mathcal{O}(1)$.
- $|f_{NL}| \gg 1$ possible also in other models!
- Large NG simply because interactions are large $|g| \gg 1$, the novelty is that this doesn't spoil inflation.

Conclusions

- ⋆ Possible to find inflationary solutions in string motivated nonlocal field theories.
- ⋆ Nonlocal theories with infinitely many derivatives can evade difficulties such as ghosts, Ostrogradski instabilities.
- ⋆ The nonlocal structure of the theory leads to novels effects:
  - Slow roll dynamics despite an extremely steep potential.
  - Possibility of large nongaussian signature in the CMB.
- ⋆ Since the p-adic string is not construed as realistic by itself, not clear how natural this is.
- ⋆ Much more work required for a fully realistic construction.

Future Directions

- ⋆ Rigorous formulation of cosmological perturbation theory, including metric perturbations, in nonlocal models.
  - – Cause for concern: ζ does not seem to freeze out on large scales.
  - – However: an exactly analogous calculation in local theory reproduces correct answer up to factors order unity.
- ⋆ Rigorous treatment of the initial value problem for differential equations with infinitely many derivatives.[a]
- ⋆ Possibility of realizing similar phenomena in more realistic string constructions.
  - – Perhaps inflation does not require flat potentials!

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[a] NB & Kamran, work in progress.