Primordial non-Gaussianity from inflation

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Trispectrum: astro-ph/0611075,

Life beyond the Gaussian, KICP, Chicago, 6th June 2007
Motivation

• Lots of models of inflation, need to predict observables
• Non-Gaussianity, observations improving rapidly
• Not just $f_{NL}$ parameterises bispectrum
• ACT, Planck, can observe/constrain trispectrum
• 2 observable parameters
• What about higher order statistics?
• Or loop corrections?

• Diagrammatic method
  • Calculates the n-point function of the primordial curvature perturbation, at tree or loop level
• Separate universe approach
• Valid for multiple fields and to all orders in slow-roll parameters
The primordial curvature perturbation $\zeta$

Calculate using the $\delta N$ formalism (valid on super horizon scales)

Separate universe approach

\[ N = \int_{t_\ast}^{t_{\text{prim}}} H dt \]

Efoldings

\[ \zeta = \delta N = N_A \delta \phi^A + \frac{1}{2} N_{AB} \delta \phi^A \delta \phi^B + \cdots \]

Where \[ N_A = \frac{\partial N}{\partial \phi^A} \] and $\delta \phi^A$ is evaluated at Hubble-exit

Starobinsky `85; Sasaki & Stewart `96  
Lyth & Rodriguez ’05 – works to any order
• Find the primordial n-point function of curvature perturbation in terms of derivatives of \( N \) and n-point function of the fields at Hubble exit

\[
\langle \delta \phi^A_{k_1} \delta \phi^B_{k_2} \delta \phi^C_{k_3} \rangle = B^{ABC}(k_1, k_2, k_3)(2\pi)^3 \delta^3(k_1 + k_2 + k_3), \\
\langle \delta \phi^A_{k_1} \delta \phi^B_{k_2} \delta \phi^C_{k_3} \delta \phi^D_{k_4} \rangle_c = T^{ABCD}(k_1, k_2, k_3, k_4)(2\pi)^3 \delta^3(k_1 + k_2 + k_3 + k_4).
\]

Maldacena (2001); Seery & Lidsey (2005)

\[ B_\zeta(k_1, k_2, k_3) = N_A N_B N_C B^{ABC}(k_1, k_2, k_3) + N_A N_B N^{AB}(P(k_1)P(k_2) + 2 \text{ perms}) \]

• The first term is unobservably small in slow roll inflation
• Often assume Gaussian, only need 2-point function
• Not if non-standard kinetic term, break in the potential...

\[
\langle \delta \phi^A_{k_1} \delta \phi^B_{k_2} \rangle = C^{AB}(k)(2\pi)^3 \delta^3(k_1 + k_2), \\
C^{AB}(k) \simeq \delta^{AB}P(k)
\]

\[ \mathcal{P}(k) = \frac{4\pi k^3}{(2\pi)^3} P(k) = \left( \frac{H_*}{2\pi} \right)^2 \]

• Work to leading order in slow roll inflation, fields are Gaussian, primordial perturbations are not Gaussian

\[
\zeta^{(1)} = N_A \delta \phi^{(1)} A \\
\zeta^{(2)} = N_A \delta \phi^{(2)} A + N_{AB} \delta \phi^{(1)} A \delta \phi^{(1)} B
\]
Diagrams from Gaussian initial fields

Here for Fourier space, can also give for real space
Rule for n-point function, at r-th order, r=n-1 is tree level

1. Draw all distinct connected diagrams with n-external lines (solid) and r propagators (dashed)
2. Assign momenta to all lines
3. Assign the appropriate factor to each vertex and propagator

\[(2\pi)^3 \delta^3(k-q_1-q_2q_3)N_{ABC}\]

4. Integrate over undetermined loop momenta
5. Divide by numerical factor (1 for all tree level terms)
6. Add all distinct permutations of the diagrams
Explicit example of the rules

For 3-point function at tree level, defined by
\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \equiv B_\zeta(k_1, k_2, k_3)(2\pi)^3\delta^3(k_1 + k_2 + k_3).
\]

There is only one diagram at tree level
\[
N_A N_D N_{BC} \delta^{AB} \delta^{CD} (2\pi)^3 \int d^3 q_1 d^3 q_2 P(q_1) P(q_2) \delta^3(k_1 - q_1 - q_2) \delta^3(k_2 + q_1) \delta^3(k_3 + q_2)
\]

After integrating the internal momentum and adding distinct permutations of the external momenta we find
\[
B_\zeta^{\text{tree}}(k_1, k_2, k_3) = N_A N_B N^{AB} (P(k_2) P(k_3) + 2 \text{ perms})
\]
Power spectrum

\[ \langle \zeta_{k_1} \zeta_{k_2} \rangle \equiv P_\zeta(k) (2\pi)^3 \delta^3(k_1 + k_2) \]

These are all of the diagrams up to 2 loop level

\[ P^\text{tree}_\zeta(k) = N_A N_A^A P(k) \]

\[ P^1_\zeta^\text{loop}(k) = \frac{1}{(2\pi)^3} \int d^3 q \left( \frac{1}{2} N_{AB} N_{AB} P(q) P(|k_1 - q|) + N_A N_B^A P(k) P(q) \right) \]

\[ P^2_\zeta^\text{loop} = \frac{1}{(2\pi)^6} \int d^3 q_1 d^3 q_2 \left( \frac{1}{4} N_{ABC} N_{ABC} P(q_1) P(q_2) P(k) + \frac{1}{4} N_A N_B^A N_C^B P(k) P(q_1) P(q_2) \right. \]

\[ \left. + \frac{1}{2} N_{ABC} N_{ABC} P(q_1) P(q_2) P(|k_1 - q_2|) + \frac{1}{6} N_{ABC} N_{ABC} N_{ABC} P(q_1) P(|q_2 - q_1|) P(|k_1 - q_2|) \right) \]
Renormalisation

• There is a way to absorb all diagrams with dressed vertices, this deals with some of the divergent terms
• A physical interpretation is work in progress
• We replace derivatives of N evaluated for the background field $\phi_0$ to the ensemble average at a general point $\phi(x)$
• Renormalised vertex = Sum of dressed vertices

\[
\langle \tilde{N}_A | \phi(x) \rangle = N_A + \frac{1}{2} N_{AB}^B \langle \delta \phi^2 \rangle + \frac{1}{8} N_{ABC}^{BC} \langle \delta \phi^2 \rangle^2 + \cdots
\]

where \[ \langle \delta \phi^2 \rangle = \int d^3q P(q) \approx \mathcal{P} \int \frac{dq}{q} \]

• The variance of phi has a log divergence for large and small scales, we remove these terms by renormalising the vertices as shown in the diagram.
Power spectrum, with renormalised vertices

The 7 diagrams shown previously, for terms with up to 2 loops reduces to 3 diagrams (in fact there is only 1 diagram at every loop level).

\[ P_{\zeta}^{\text{tree}} = \langle \tilde{N}_A \rangle \langle \tilde{N}^A \rangle P(k) \]
\[ P_{\zeta}^{1\text{ loop}} = \frac{1}{2} \langle \tilde{N}_{AB} \rangle \langle \tilde{N}^{AB} \rangle \frac{1}{(2\pi)^3} \int d^3q P(q) P(|k_1 - q|) \]
\[ P_{\zeta}^{2\text{ loop}} = \frac{1}{3!} \frac{1}{(2\pi)^6} \int d^3q_1 d^3q_2 \langle \tilde{N}_{AB} \rangle \langle \tilde{N}^{ABC} \rangle P(q_1) P(|q_2 - q_1|) P(|q_2 - k_1|) \]
Extension to non-Gaussian fields

We can extend the previous diagrammatic rules to fields with a non-Gaussian distribution at Hubble exit, also to all orders in slow roll. We have to include terms like

\[
\langle \delta \phi_{k_1}^A \delta \phi_{k_2}^B \rangle = C^{AB}(k)(2\pi)^3 \delta^3(k_1 + k_2),
\]
\[
\langle \delta \phi_{k_1}^A \delta \phi_{k_2}^B \delta \phi_{k_3}^C \rangle = B^{ABC}(k_1, k_2, k_3)(2\pi)^3 \delta^3(k_1 + k_2 + k_3),
\]
\[
\langle \delta \phi_{k_1}^A \delta \phi_{k_2}^B \delta \phi_{k_3}^C \delta \phi_{k_4}^D \rangle_C = T^{ABCD}(k_1, k_2, k_3, k_4)(2\pi)^3 \delta^3(k_1 + k_2 + k_3 + k_4).
\]

The diagrams now have vertices of internal (dashed) legs, corresponding to 3- and higher-point functions of the fields

\[C^{AB}(q) \sim \delta^{AB} P(q)\]

CB and Wands, 2006
Application to the trispectrum

\[ \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle_c \equiv T_\zeta(k_1, k_2, k_3, k_4)(2\pi)^3 \delta^3(k_1 + k_2 + k_3 + k_4). \]

\[ T_\zeta^{trc}(k_1, k_2, k_3, k_4) = N_A N_B N_C N_D T^{ABCD}(k_1, k_2, k_3, k_4) \]
\[ + N_{AB} N_{CD} N_{EF} \left( C^{AC}(k_1) B^{BDE}(k_1 + k_2, k_3, k_4) + (11 \text{ perms}) \right) \]
\[ + N_{AB} N_{CD} N_{EF} \left( C^{AC}(k_1 + k_3) C^{BD}(k_3) C^{DF}(k_4) + (11 \text{ perms}) \right) \]
\[ + N_{AB} N_{CD} N_{EF} \left( C^{AC}(k_2) C^{BE}(k_3) C^{CF}(k_4) + (3 \text{ perms}) \right). \]

Seery & Lidsey, 2006
Byrnes, Sasaki & Wands, 2006

Note that the two last terms are non-zero even for Gaussian fields
Observable parameters, bispectrum and trispectrum

Assuming slow-roll inflation, fields Gaussian and work at leading order in slow roll.
We define 3 k independent non-linearity parameters

\[ B_\zeta(k_1, k_2, k_3) = \frac{6}{5} f_{NL} \left[ P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1) \right] \]

\[ T_\zeta(k_1, k_2, k_3, k_4) = \tau_{NL} \left[ P_\zeta(|k_1 + k_3|)P_\zeta(k_3)P_\zeta(k_4) + (11 \text{ perms}) \right] + \frac{54}{25} g_{NL} \left[ P_\zeta(k_2)P_\zeta(k_3)P_\zeta(k_4) + (3 \text{ perms}) \right] \]

Note that \( \tau_{NL} \) and \( g_{NL} \) both appear at leading order in the trispectrum
The coefficients have a different k dependence, \( P_\zeta \propto k^{-3} \)
\[ B^\text{tree}_\zeta(k_1, k_2, k_3) = N_A N_B N^{AB} (P(k_2) P(k_3) + 2 \text{ perms}) \]

\[ T^\text{tree}_\zeta(k_1, k_2, k_3, k_4) = N_A B_N C N^A N^B N^C (P(|k_1 + k_3|) P(k_3) P(k_4) + (11 \text{ perms})) + N_A B_C N^A N^B N^C (P(k_2) P(k_3) P(k_4) + (3 \text{ perms})) \, . \]

Hence the non-linearity parameters are

\[ f_{NL} = \frac{5 N_A N_B N^{AB}}{6 (N^C N^C)^2} \]

\[ \tau_{NL} = \frac{N_A B N^A C N^B N_C}{(N_D N^D)^3} \quad g_{NL} = \frac{25 N_A B C N^A N^B N_C}{54 (N_D N^D)^3} \]
Single field inflation

Specialise to the case where one field generates the primordial curvature perturbation
Includes many of the cases considered in the literature:
• Standard single field inflation
• Curvaton scenario
• Modulated reheating

\[ f_{NL} = \frac{5}{6} \frac{N'''}{(N')}^2, \quad g_{NL} = \frac{25}{54} \frac{N'''}{(N')}^3, \quad \tau_{NL} = \frac{(N''')}{(N')}^4 = \frac{36}{25} f_{NL}^2 \]

where \[ N' = \frac{\partial N}{\partial \phi} \]

2 independent parameters
Consistency condition between bispectrum and 1 term of the trispectrum

Simple relation between non-linearity parameters and zeta

\[ \zeta = \zeta_1 + \frac{3}{5} f_{NL} \zeta_1^2 + \frac{9}{25} g_{NL} \zeta_1^3 + \cdots \]
Non-Gaussianity from slow-roll inflation?

**single inflaton field**
- can evaluate non-Gaussianity at Hubble exit (zeta is conserved)
  \[ f_{NL} = \frac{5}{6} \frac{N''}{(N')^2} = \frac{5}{6} (\eta - 2\epsilon) \quad g_{NL} = \frac{25}{54} \frac{N'''}{(N')^3} = \frac{25}{54} (2\epsilon\eta - 2\eta^2 + \xi^2) \]
- **undetectable** with the CMB

**multiple field inflation**
- difficult to get large non-Gaussianity during inflation
  No explicit model has been constructed

Easier to generate non-Gaussianity after inflation
E.g. Curvaton, modulated (p)reheating, inhomogeneous end of inflation

**Curvaton scenario**
The primordial curvature perturbation is generated from a curvaton field that is subdominant during inflation
If the ratio of the curvaton’s energy density to the total energy density is small
\[ r = \left[ \frac{3\rho_X}{3\rho_X + 4\rho_r} \right]_{\text{decay}} \ll 1 \]
Non-linearity parameters are large
\[ f_{NL} = \mathcal{O}(1/r), \quad g_{NL} = \mathcal{O}(1/r^2) \]
Sasaki, Valiviita & Wands 2006
Observational constraints

WMAP3 bound on the bispectrum

\[-54 < f_{NL} < 114\]

Bound on the trispectrum?
Only an “indirect bound”

\[|\tau_{NL}| < 10^4\] \hspace{1cm} \text{Lyth ‘06}

No bound on \(g_{NL}\)

Plans to constrain the trispectrum in the near future, WMAP, ACT and Planck

Assuming no detection, Planck is predicted to reach

\[|f_{NL}| < 3, \quad |\tau_{NL}| < 560\] \hspace{1cm} \text{Kogo and Komatsu ‘06}
Conclusions

• We have presented a diagrammatic approach to calculating n-point function including loop corrections at any order
• Valid for non-Gaussian fields and to all orders in slow roll
• Trispectrum has 2 observable parameters
  - only in single field inflation $\tau_{NL} \propto f_{NL}^2$
Curvaton scenario

• In the curvaton scenario the primordial curvature perturbation is generated from a scalar field that is light and subdominant during inflation but becomes a significant proportion of the energy density of the universe sometime after inflation.

• The energy density of the curvaton is a function of the field value at Hubble-exit

\[ \rho_\chi \propto g^2(\chi_*) \]

• The ratio of the curvaton’s energy density to the total energy density is

\[ r = \left[ \frac{3\rho_\chi}{3\rho_\chi + 4\rho_r} \right]_{\text{decay}} \]
Curvaton scenario cont.

- In the case that $r \ll 1$
- The non-linearity parameters are given by

\[
\begin{align*}
    f_{NL} & \simeq \frac{5}{4r} \left( 1 + \frac{gg''}{g'^2} \right), \\
    g_{NL} & \simeq \frac{25}{24r^2} \left( \frac{g^2 g'''}{g'^3} + 3 \frac{gg''}{g'^2} \right). \\
\end{align*}
\]

Sasaki, Valiviita and Wands 2006

- In general this generates a large bispectrum and trispectrum.

- If \( gg'' / g'^2 \sim -1 \) the bispectrum will be small

In this case the first non-Gaussianity signal might come from the trispectrum through \( g_{NL} \gg 1 \).

Enqvist and Nurmi, 2005
defining the primordial density perturbation

gauge-dependent density perturbation, $\delta \rho$, and spatial curvature, $\psi$

gauge-invariant combination:

dimensionless density perturbation on spatially flat hypersurfaces

$$\zeta = - \frac{H}{\dot{\rho}} \delta \rho - \psi$$

can be constant on large scales for adiabatic perturbations

Wands, Malik, Lyth & Liddle (2000)
primordial perturbations from scalar fields

in radiation-dominated era
curvature perturbation $\zeta$ on
uniform-density hypersurface

during inflation
field perturbations $\phi(x,t)$ on
initial spatially-flat hypersurface

on large scales, neglect spatial gradients, treat as “separate universes”

the $\delta N$ formalism

$$\zeta = \delta N = N(\phi_{\text{initial}}) - \bar{N} \approx \sum_i \frac{\partial N}{\partial \phi_i} \delta \phi_i$$

Starobinsky `85; Sasaki & Stewart `96
Lyth & Rodriguez `05 – works to any order