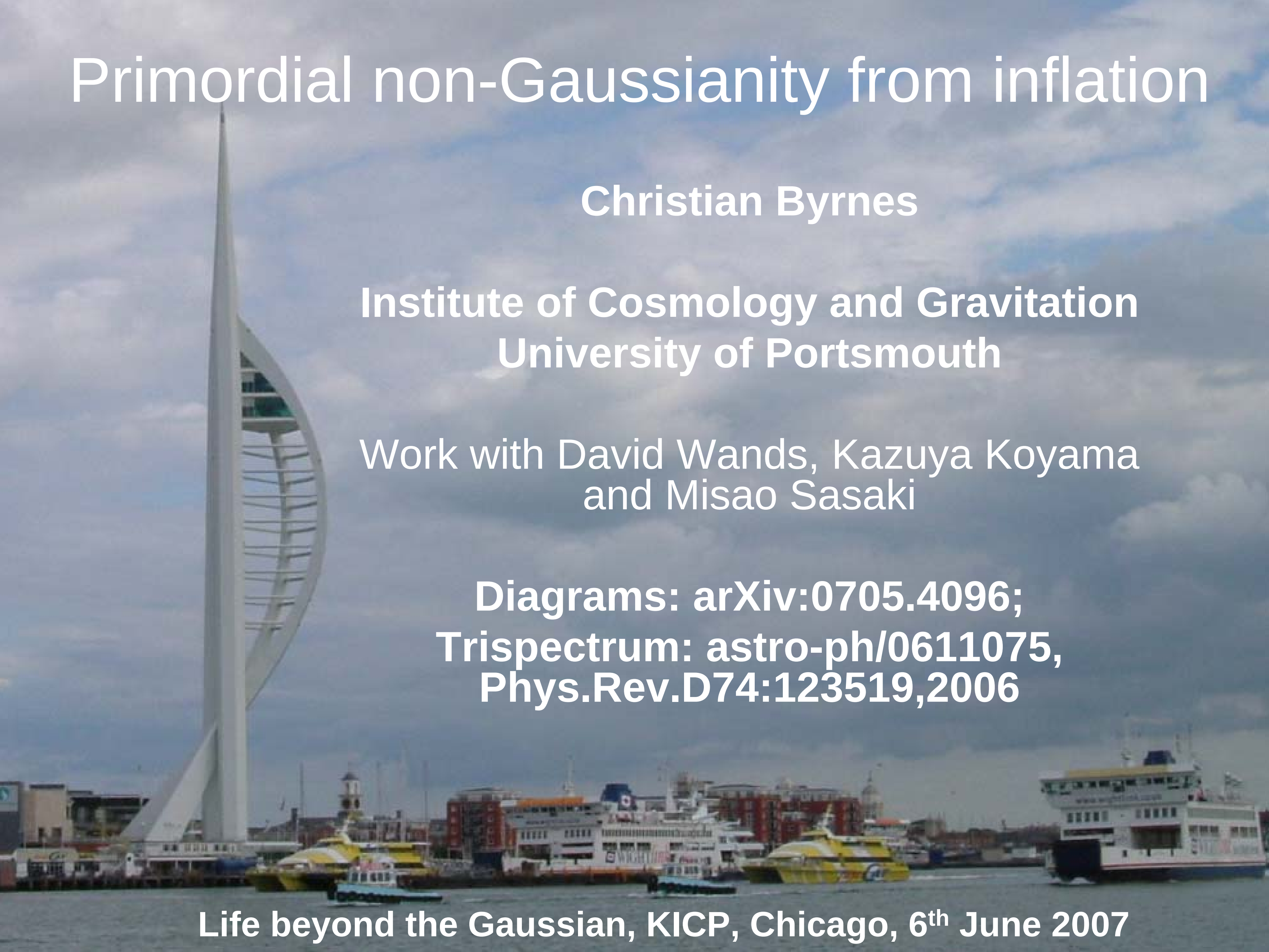

Primordial non-Gaussianity from inflation

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Work with David Wands, Kazuya Koyama
and Misao Sasaki

Diagrams: arXiv:0705.4096;
Trispectrum: astro-ph/0611075,
Phys.Rev.D74:123519,2006

Life beyond the Gaussian, KICP, Chicago, 6th June 2007

Motivation

- Lots of models of inflation, need to predict observables
- Non-Gaussianity, observations improving rapidly
- Not just f_{NL} parameterises bispectrum
- ACT, Planck, can observe/constrain trispectrum
- 2 observable parameters
- What about higher order statistics?
- Or loop corrections?

- Diagrammatic method
- Calculates the n-point function of the primordial curvature perturbation, at tree or loop level
- Separate universe approach
- Valid for multiple fields and to all orders in slow-roll parameters

The primordial curvature perturbation ζ

Calculate using the δN formalism (valid on super horizon scales)

Starobinsky `85; Sasaki & Stewart `96
Lyth & Rodriguez ’05 – works to any order

Separate universe approach

$$N = \int_{t_*}^{prim} H dt$$

Efoldings

$$\zeta = \delta N = N_A \delta\phi^A + \frac{1}{2} N_{AB} \delta\phi^A \delta\phi^B + \cdots$$

Where $N_A \equiv \frac{\partial N}{\partial \phi^A}$ and $\delta\phi^A$ is evaluated at Hubble-exit

- Find the primordial n-point function of curvature perturbation in terms of derivatives of N and n-point function of the fields at Hubble exit

Maldacena (2001); Seery & Lidsey (2005)

$$\begin{aligned} \langle \delta\phi^A_{\mathbf{k}_1}\,\delta\phi^B_{\mathbf{k}_2}\,\delta\phi^C_{\mathbf{k}_3}\rangle &= B^{ABC}(k_1,k_2,k_3)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,, \\ \langle \delta\phi^A_{\mathbf{k}_1}\,\delta\phi^B_{\mathbf{k}_2}\,\delta\phi^C_{\mathbf{k}_3}\,\delta\phi^D_{\mathbf{k}_4}\rangle_c &= T^{ABCD}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\,. \end{aligned}$$

Seery, Lidsey & Sloth (2006)

$$B_\zeta(k_1,k_2,k_3) = N_A N_B N_C B^{ABC}(k_1,k_2,k_3) + N_A N_B N^{AB}\left(P(k_1)P(k_2) + 2\text{ perms}\right)$$

- The first term is unobservably small in slow roll inflation
- Often assume Gaussian, only need 2-point function
- Not if non-standard kinetic term, break in the potential…

$$\langle \delta\phi^A_{\mathbf{k}_1}\,\delta\phi^B_{\mathbf{k}_2}\rangle = C^{AB}(k)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}), \qquad C^{AB}(k) \simeq \delta^{AB}P(k)$$

$$\mathcal{P}(k) = \frac{4\pi k^3}{(2\pi)^3}P(k) = \left(\frac{H_*}{2\pi}\right)^2$$

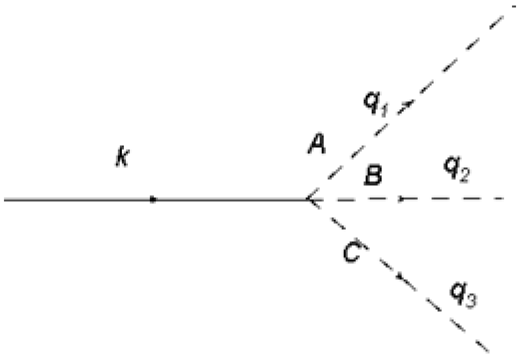
- Work to leading order in slow roll inflation, fields are Gaussian, primordial perturbations are not Gaussian

$$\zeta^{(1)} = N_A\delta\phi^{(1)A} \qquad \zeta^{(2)} = N_A\delta\phi^{(2)A} + N_{AB}\delta\phi^{(1)A}\delta\phi^{(1)B}$$

Diagrams from Gaussian initial fields

Here for Fourier space, can also give for real space
Rule for n-point function, at r-th order, r=n-1 is tree level

1. Draw all distinct connected diagrams with n-external lines (solid) and r propagators (dashed)
2. Assign momenta to all lines
3. Assign the appropriate factor to each vertex and propagator



$(2\pi)^3\delta^3(k-q_1-q_2-q_3)N_{ABC}$

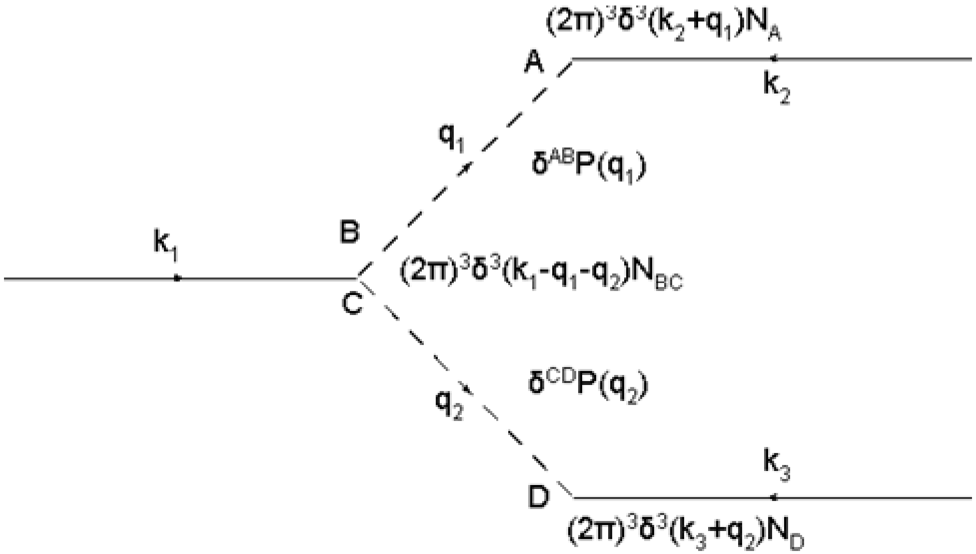
$\delta^{AB}P(q)$

4. Integrate over undetermined loop momenta
5. Divide by numerical factor (1 for all tree level terms)
6. Add all distinct permutations of the diagrams

Explicit example of the rules

For 3-point function at tree level, defined by

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle \equiv B_\zeta(k_1,k_2,k_3)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,.$$



There is only one diagram at tree level

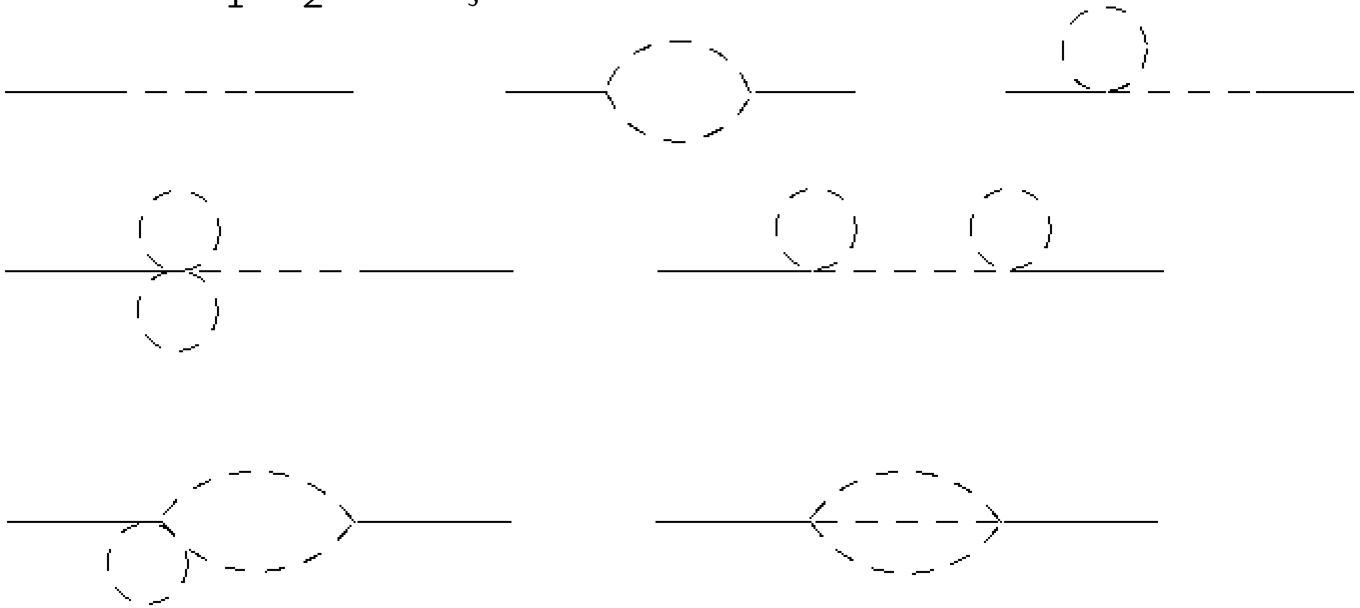
$$N_A N_D N_{BC}\delta^{AB}\delta^{CD}(2\pi)^3\int d^3q_1 d^3q_2 P(q_1)P(q_2)\delta^3(\mathbf{k_1}-\mathbf{q_1}-\mathbf{q_2})\delta^3(\mathbf{k_2}+\mathbf{q_1})\delta^3(\mathbf{k_3}+\mathbf{q_2})$$

After integrating the internal momentum and adding distinct permutations of the external momenta we find

$$B_\zeta^{\text{tree}}(k_1,k_2,k_3) = N_A N_B N^{AB}\left(P(k_2)P(k_3) + 2\text{ perms}\right)$$

Power spectrum

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle \equiv P_\zeta(k)(2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2)$$



These are all of the diagrams up to 2 loop level

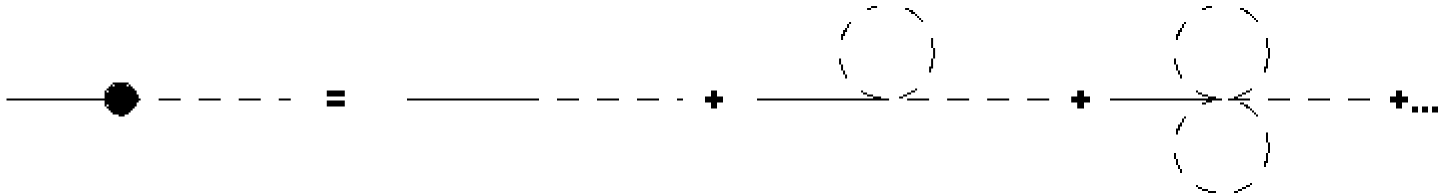
$$P_\zeta^{\text{tree}}(k) = N_A N^A P(k)$$

$$P_\zeta^{\text{1 loop}}(k) = \frac{1}{(2\pi)^3} \int d^3q \left(\frac{1}{2} N_{AB} N^{AB} P(q) P(|\mathbf{k}_1 - \mathbf{q}|) + N_A N^{AB}_B P(k) P(q) \right)$$

$$\begin{aligned} P_\zeta^{\text{2 loop}} \quad &= \quad \frac{1}{(2\pi)^6} \int d^3q_1 d^3q_2 \left(\frac{1}{4} N^{AB}_{ABC} N^C P(q_1) P(q_2) P(k) + \frac{1}{4} N^A_{AB} N^{BC}_C P(k) P(q_1) P(q_2) \right. \\ &\left. + \frac{1}{2} N_{AB} N^{ABC}_C P(q_1) P(q_2) P(|\mathbf{k}_1 - \mathbf{q}_2|) + \frac{1}{6} N_{ABC} N^{ABC} P(q_1) P(|\mathbf{q}_2 - \mathbf{q}_1|) P(|\mathbf{k}_1 - \mathbf{q}_2|) \right) \end{aligned}$$

Renormalisation

- There is a way to absorb all diagrams with dressed vertices, this deals with some of the divergent terms
- A physical interpretation is work in progress
- We replace derivatives of N evaluated for the background field ϕ_0 to the ensemble average at a general point $\phi(\mathbf{x})$
- Renormalised vertex = Sum of dressed vertices



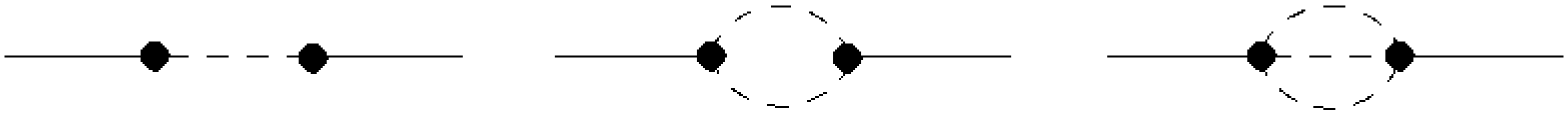
$$\langle \tilde{N}_A|_{\phi(\mathbf{x})}\rangle = N_A + \frac{1}{2}N_{AB}^{B}\langle\delta\phi^2\rangle + \frac{1}{8}N_{ABC}^{BC}\langle\delta\phi^2\rangle^2 + \cdots$$

where $$\langle\delta\phi^2\rangle = \int d^3q P(q) \simeq \mathcal{P}\int\frac{dq}{q}$$

- The variance of phi has a log divergence for large and small scales, we remove these terms by renormalising the vertices as shown in the diagram.

Power spectrum, with renormalised vertices

The 7 diagrams shown previously, for terms with up to 2 loops reduces to 3 diagrams (in fact there is only 1 diagram at every loop level).



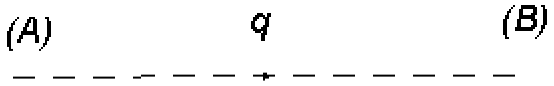
$$
\begin{aligned}
P_\zeta^{\text{tree}} &= \langle \tilde{N}_A \rangle \langle \tilde{N}^A \rangle P(k) \\
P_\zeta^{\text{1 loop}} &= \frac{1}{2} \langle \tilde{N}_{AB} \rangle \langle \tilde{N}^{AB} \rangle \frac{1}{(2\pi)^3} \int d^3q P(q) P(|\mathbf{k}_1 - \mathbf{q}|) \\
P_\zeta^{\text{2 loop}} &= \frac{1}{3!} \frac{1}{(2\pi)^6} \int d^3q_1 d^3q_2 \langle \tilde{N}_{ABC} \rangle \langle \tilde{N}^{ABC} \rangle P(q_1) P(|\mathbf{q}_2 - \mathbf{q}_1|) P(|\mathbf{q}_2 - \mathbf{k}_1|)
\end{aligned}
$$

Extension to non-Gaussian fields

We can extend the previous diagrammatic rules to fields with a non-Gaussian distribution at Hubble exit, also to all orders in slow roll.
We have to include terms like

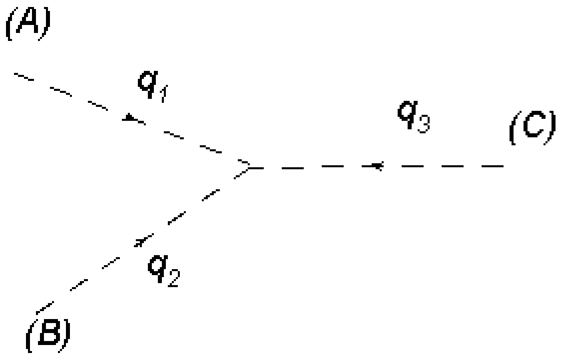
$$\begin{aligned}
\langle \delta\phi^A_{\mathbf{k}_1}\,\delta\phi^B_{\mathbf{k}_2}\rangle &= C^{AB}(k)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2})\,,\\
\langle \delta\phi^A_{\mathbf{k}_1}\,\delta\phi^B_{\mathbf{k}_2}\,\delta\phi^C_{\mathbf{k}_3}\rangle &= B^{ABC}(k_1,k_2,k_3)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,,\\
\langle \delta\phi^A_{\mathbf{k}_1}\,\delta\phi^B_{\mathbf{k}_2}\,\delta\phi^C_{\mathbf{k}_3}\,\delta\phi^D_{\mathbf{k}_4}\rangle_c &= T^{ABCD}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)(2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\,.
\end{aligned}$$

The diagrams now have vertices of internal (dashed) legs, corresponding to 3- and higher-point functions of the fields



$C^{AB}(q) \sim \delta^{AB} P(q)$

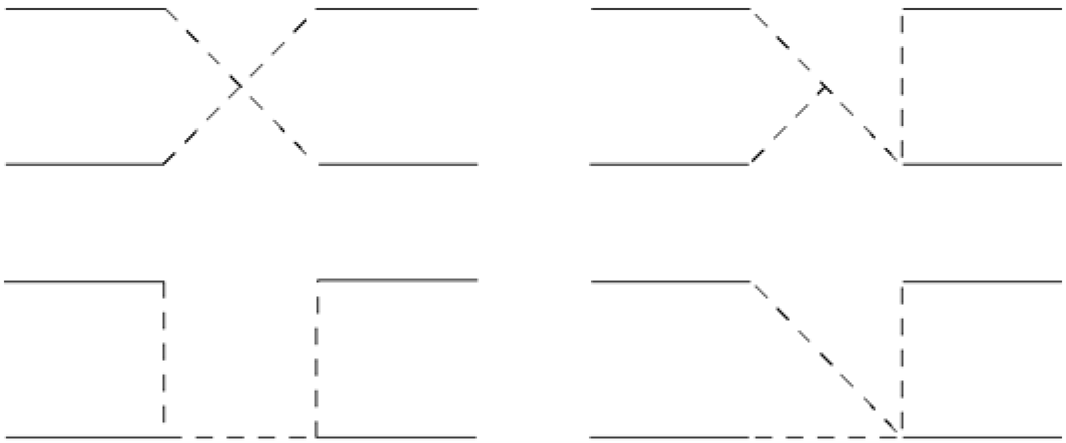
CB and Wands, 2006



$(2\pi)^3\delta^3(q_1+q_2+q_3)B^{ABC}(q_1,q_2,q_3)$

Application to the trispectrum

$$\langle\zeta_{\mathbf{k}_1}\,\zeta_{\mathbf{k}_2}\,\zeta_{\mathbf{k}_3}\,\zeta_{\mathbf{k}_4}\rangle_c \equiv T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\,.$$



$$\begin{aligned}
T_\zeta^{\text{tree}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) \;=\;& N_A N_B N_C N_D T^{ABCD}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\\
&+N_{AB}N_C N_D N_E\left(C^{AC}(k_1)B^{BDE}(|\mathbf{k}_1+\mathbf{k}_2|,k_3,k_4)+(11\text{ perms})\right)\\
&+N_{AB}N_{CD}N_E N_F\left(C^{AC}(|\mathbf{k}_1+\mathbf{k}_3|)C^{BD}(k_3)C^{DF}(k_4)+(11\text{ perms})\right)\\
&+N_{ABC}N_D N_E N_F\left(C^{AD}(k_2)C^{BE}(k_3)C^{CF}(k_4)+(3\text{ perms})\right)\,.
\end{aligned}$$

Seery & Lidsey, 2006
Byrnes, Sasaki & Wands, 2006

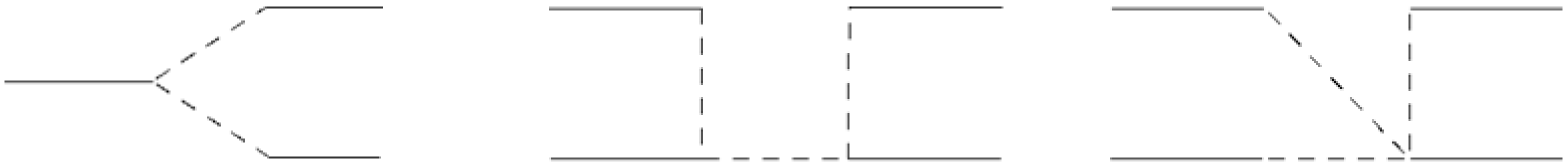
Note that the two last terms are non-zero even for Gaussian fields

Observable parameters, bispectrum and trispectrum

Assuming slow-roll inflation, fields Gaussian and work at leading order in slow roll.
We define 3 k independent non-linearity parameters

$$
\begin{aligned}
B_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \frac{6}{5} f_{NL}\left[P_\zeta(k_1)P_\zeta(k_2)+P_\zeta(k_2)P_\zeta(k_3)+P_\zeta(k_3)P_\zeta(k_1)\right] \\
T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= \tau_{NL}\left[P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)P_\zeta(k_3)P_\zeta(k_4)+(11\text{ perms})\right] \\
&\quad +\frac{54}{25} g_{NL}\left[P_\zeta(k_2)P_\zeta(k_3)P_\zeta(k_4)+(3\text{ perms})\right]
\end{aligned}
$$

Note that τ_{NL} and g_{NL} both appear at leading order in the trispectrum
The coefficients have a different k dependence, $P_\zeta \propto k^{-3}$



$$B_\zeta^{\mathrm{tree}}(k_1,k_2,k_3) = N_A N_B N^{AB}\left(P(k_2)P(k_3) + 2\text{ perms}\right)$$

$$\begin{aligned} T_\zeta^{\mathrm{tree}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= N_{AB}N^B_C N^A N^C \left(P(|\mathbf{k}_1+\mathbf{k}_3|)P(k_3)P(k_4) + (11\text{ perms})\right) \\ &\quad + N_{ABC}N^A N^B N^C \left(P(k_2)P(k_3)P(k_4) + (3\text{ perms})\right) . \end{aligned}$$

Hence the non-linearity parameters are

$$f_{NL} = \frac{5}{6}\frac{N_A N_B N^{AB}}{\left(N_C N^C\right)^2}$$

$$\tau_{NL} = \frac{N_{AB}N^{AC}N^B N_C}{(N_D N^D)^3} \qquad g_{NL} = \frac{25}{54}\frac{N_{ABC}N^A N^B N^C}{(N_D N^D)^3}$$

Single field inflation

Specialise to the case where one field generates the primordial curvature perturbation

Includes many of the cases considered in the literature:

- Standard single field inflation
- Curvaton scenario
- Modulated reheating

$$f_{NL} = \frac{5}{6}\frac{N''}{(N')^2}, \qquad g_{NL} = \frac{25}{54}\frac{N'''}{(N')^3}, \qquad \tau_{NL} = \frac{(N'')^2}{(N')^4} = \frac{36}{25}{f_{NL}}^2$$

where $N' = \frac{\partial N}{\partial \phi}$

2 independent parameters

Consistency condition between bispectrum and 1 term of the trispectrum

Simple relation between non-linearity parameters and zeta

$$\zeta = \zeta_1 + \frac{3}{5}f_{NL}\zeta_1^2 + \frac{9}{25}g_{NL}\zeta_1^3 + \cdots$$

Non-Gaussianity from slow-roll inflation?

single inflaton field

- can evaluate non-Gaussianity at Hubble exit (zeta is conserved)

$$f_{NL} = \frac{5}{6}\frac{N''}{(N')^2} = \frac{5}{6}(\eta - 2\epsilon) \quad g_{NL} = \frac{25}{54}\frac{N'''}{(N')^3} = \frac{25}{54}\left(2\epsilon\eta - 2\eta^2 + \xi^2\right)$$

- **undetectable** with the CMB

multiple field inflation Vernizzi & Wands 06, Battefeld & Easther 06

- difficult to get large non-Gaussianity during inflation

  No explicit model has been constructed

Easier to generate non-Gaussianity after inflation
E.g. Curvaton, modulated (p)reheating, inhomogeneous end of inflation

Curvaton scenario

The primordial curvature perturbation is generated from a curvaton field that is subdominant during inflation

If the ratio of the curvaton's energy density to the total energy density is small $r = \left[\frac{3\rho_\chi}{3\rho_\chi + 4\rho_r}\right]_{\text{decay}} \ll 1$

Non-linearity parameters are large $f_{NL} = \mathcal{O}(1/r)\,, \qquad g_{NL} = \mathcal{O}(1/r^2)$

Sasaki, Valiviita & Wands 2006

Observational constraints

WMAP3 bound on the bispectrum

$$-54 < f_{NL} < 114$$

Bound on the trispectrum?
Only an “indirect bound”

$|\tau_{NL}| < 10^4$ Lyth ‘06

No bound on g_{NL}

Plans to constrain the trispectrum in the near future, WMAP, ACT and Planck

Assuming no detection, Planck is predicted to reach

$|f_{NL}| < 3,\qquad |\tau_{NL}| < 560$ Kogo and Komatsu ‘06

Conclusions

- We have presented a diagrammatic approach to calculating n-point function including loop corrections at any order
- Valid for non-Gaussian fields and to all orders in slow roll
- Trispectrum has 2 observable parameters

  - only in single field inflation $\tau_{NL} \propto {f_{NL}}^2$

Curvaton scenario

- In the curvaton scenario the primordial curvature perturbation is generated from a scalar field that is light and subdominant during inflation but becomes a significant proportion of the energy density of the universe sometime after inflation.

- The energy density of the curvaton is a function of the field value at Hubble-exit

$$\rho_\chi \propto g^2(\chi_*)$$

- The ratio of the curvaton's energy density to the total energy density is

$$r = \left[\frac{3\rho_\chi}{3\rho_\chi + 4\rho_r}\right]_{\text{decay}}$$

Curvaton scenario cont.

- In the case that r <<1
- The non-linearity parameters are given by

$$f_{NL} \simeq \frac{5}{4r}\left(1+\frac{gg''}{g'^2}\right) ,$$

$$g_{NL} \simeq \frac{25}{24r^2}\left(\frac{g^2g'''}{g'^3}+3\frac{gg''}{g'^2}\right) .$$

Sasaki, Valiviita and Wands 2006

- In general this generates a large bispectrum and trispectrum.

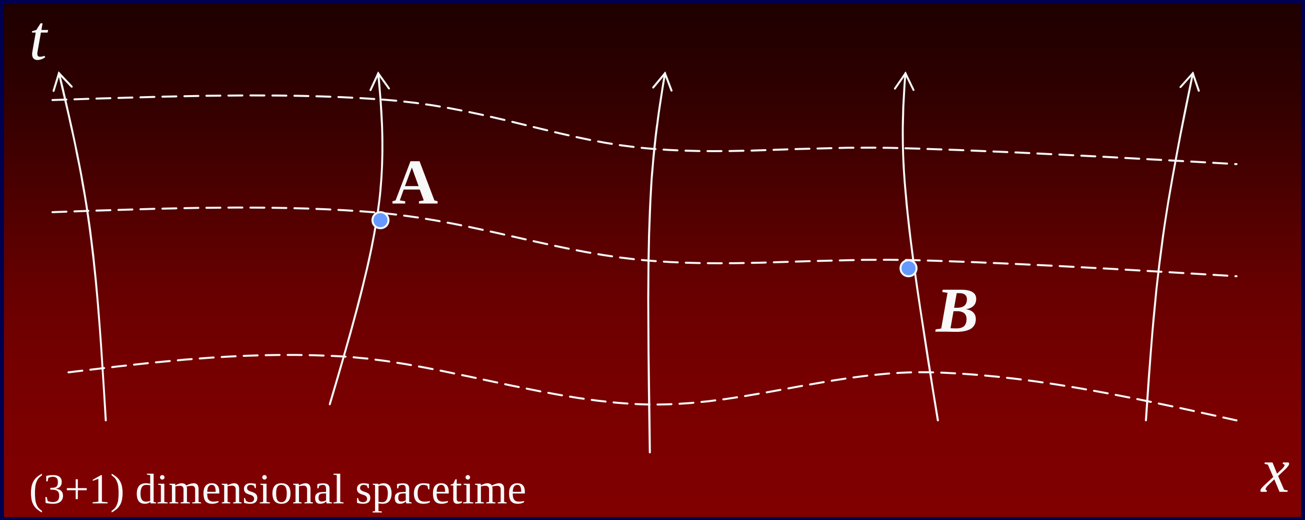
- If $gg''/g'^2 \simeq -1$ the bispectrum will be small

In this case the first non-Gaussianity signal might come from the trispectrum through $g_{NL} >> 1.$

Enqvist and Nurmi, 2005

defining the primordial density perturbation

gauge-dependent density perturbation, $\delta\rho$, and spatial curvature, ψ



(3+1) dimensional spacetime

gauge-invariant combination:

dimensionless density perturbation on spatially flat hypersurfaces

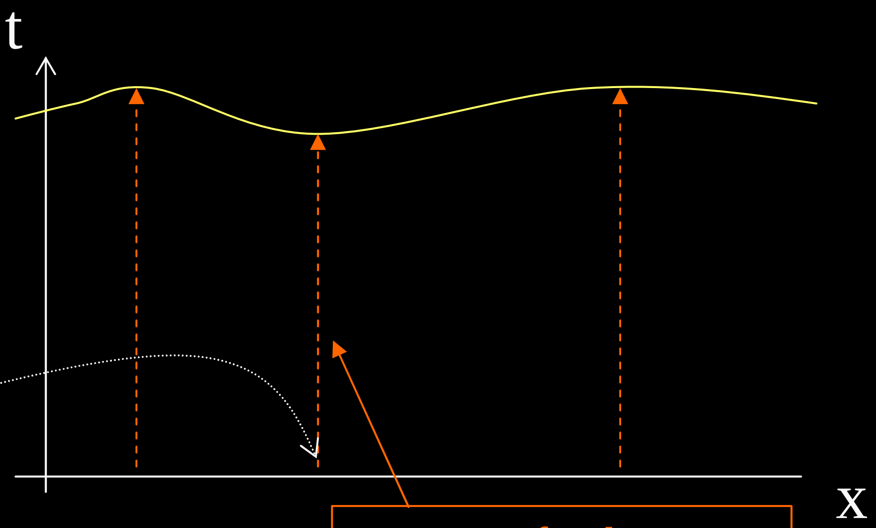
$$\zeta = -\frac{H}{\dot{\rho}}\delta\rho - \psi$$

constant on large scales for adiabatic perturbations

Wands, Malik, Lyth & Liddle (2000)

primordial perturbations from scalar fields

in radiation-dominated era
curvature perturbation ζ on
uniform-density hypersurface



during inflation
field perturbations $\phi(x,t_i)$ on
initial spatially-flat hypersurface

$$N = \int_{initial}^{final} H\, dt$$

on large scales, neglect spatial gradients, treat as “separate universes”

the δN formalism

$$\zeta = \delta N = N\left(\phi_{initial}\right) - \bar{N} \approx \sum_I \frac{\partial N}{\partial \phi_I} \delta\phi_I$$

Starobinsky `85; Sasaki & Stewart `96
Lyth & Rodriguez ’05 – works to any order