Primordial non-Gaussianity and the CMB bispectrum

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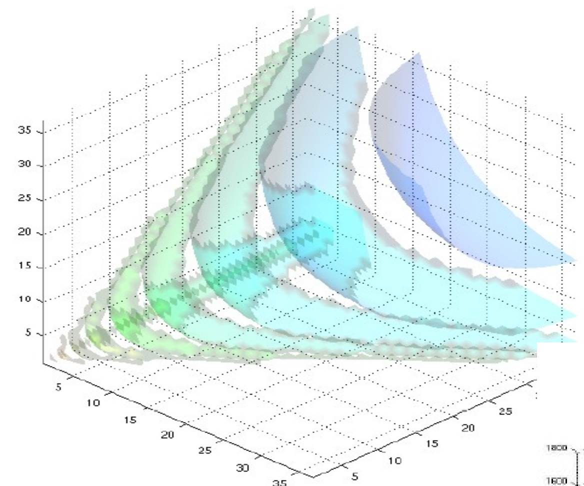
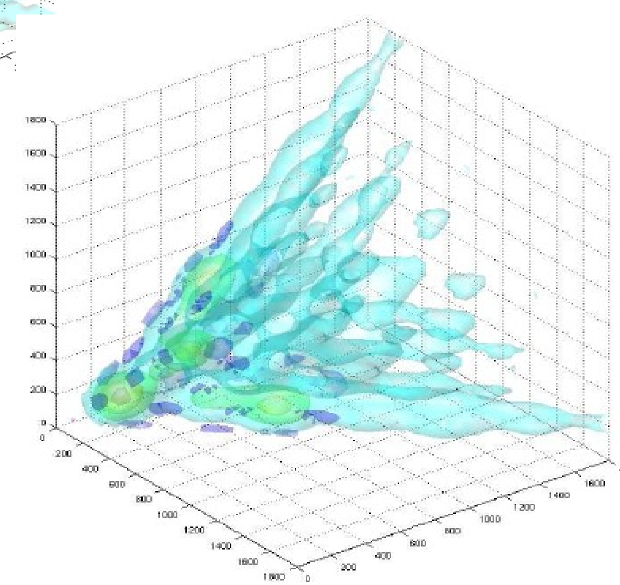
- Why Bother?
- A Difficult Integral
- Current Approach
- New Approach
- Numerical Work
- Results
- Flat Sky
- Measurement
- Conclusions





ArXiv: astro-ph/0612713

Why Bother?

- Slow roll inflation predicts a Gaussian CMB
- If the CMB is Gaussian it is completely described by the power spectrum
- Many more realistic models predict deviations from Gaussianity
- The bispectrum has been shown to be optimal for measuring this non-Gaussianity
- Essentially it is about making sure we are getting all the available information out of the CMB

A Difficult Integral

- Decompose the temperature anisotropy into multipoles

$$a_{lm} = \int d\hat{\mathbf{n}} \frac{\Delta T}{T}(\hat{\mathbf{n}}) Y_{lm}^{*}(\hat{\mathbf{n}})$$

- The temperature can also be represented in terms of the primordial gravitational potential perturbation and the radiation transfer function

$$\frac{\Delta T}{T}(\hat{\mathbf{n}}) = \int \frac{d^3 k}{(2\pi)^3} \sum_{l=0}^{\infty} (-i)^l (2l+1) \Psi(\mathbf{k}) \Delta_l(k) P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})$$

- We replace the Legendre polynomial with it's spherical harmonic expansion

$$P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{k}}) Y_{lm}^{*}(\hat{\mathbf{n}})$$

- Substituting gives an expression for the multipoles in terms of the primordial gravitational potential perturbation and the radiation transfer function

$$a_{lm} \;=\; 4\pi (-i)^l \int \frac{d^3 k}{(2\pi)^3} \Psi(k) \Delta_l(k) Y_{lm}^{*}(\hat{\mathbf{k}})$$

A Difficult Integral

- The bispectrum is the three point correlator of the a_{lm}'s
- The three point correlator of the primordial gravitational potential perturbation consists of a delta function and a shape function F which only depends on the magnitudes of the k's
- We substitute and replace the delta function with its integral representation expanded in Bessel functions and spherical harmonics
- The bispectrum then splits into a geometric factor given by the Gaunt integral times the reduced bispectrum

$$\begin{aligned} B^{m_1 m_2 m_3}_{l_1 l_2 l_3} &= \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle \\ &= (4\pi)^3 (-i)^{l_1+l_2+l_3} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{d^3\mathbf{k}_3}{(2\pi)^3} \langle \Psi(\mathbf{k}_1)\Psi(\mathbf{k}_2)\Psi(\mathbf{k}_3) \rangle \\ & \quad \Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3) Y^*_{l_1 m_1}(\hat{\mathbf{k}}_1) Y^*_{l_2 m_2}(\hat{\mathbf{k}}_2) Y^*_{l_3 m_3}(\hat{\mathbf{k}}_3). \end{aligned}$$

$$\langle \Psi(\mathbf{k}_1)\Psi(\mathbf{k}_2)\Psi(\mathbf{k}_3) \rangle = (2\pi)^3 F(k_1, k_2, k_3) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$$

$$\delta(\mathbf{k}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\cdot\mathbf{x}} d^3x.$$

$$e^{i\mathbf{k}_1\cdot\mathbf{x}} = 4\pi \sum_l i^l j_l(k_1 x) \sum_m Y_{lm}(\hat{\mathbf{k}}_1) Y^*_{lm}(\hat{\mathbf{x}})$$

$$B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3}$$

$$\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \int d\Omega\, Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int dx dk_1 dk_2 dk_3 (x k_1 k_2 k_3)^2 F(k_1, k_2, k_3) \Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3) j_{l_1}(k_1 x) j_{l_2}(k_2 x) j_{l_3}(k_3 x).$$

A Difficult Integral

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int dx dk_1 dk_2 dk_3 (x k_1 k_2 k_3)^2 F(k_1, k_2, k_3) \Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) j_{l_1}(k_1 x) j_{l_2}(k_2 x) j_{l_3}(k_3 x).$$

The reduced bispectrum is given by a 4D-integral over 6 highly oscillatory functions which is impossible to calculate on realistic time-scales

How can we proceed?

Current Approach

- What if we could separate the primordial bispectrum into the product of functions which only depend on one k?

$$F(k_1, k_2, k_3) = X(k_1)Y(k_2)Z(k_3) + 5\,\mathrm{permutations}$$

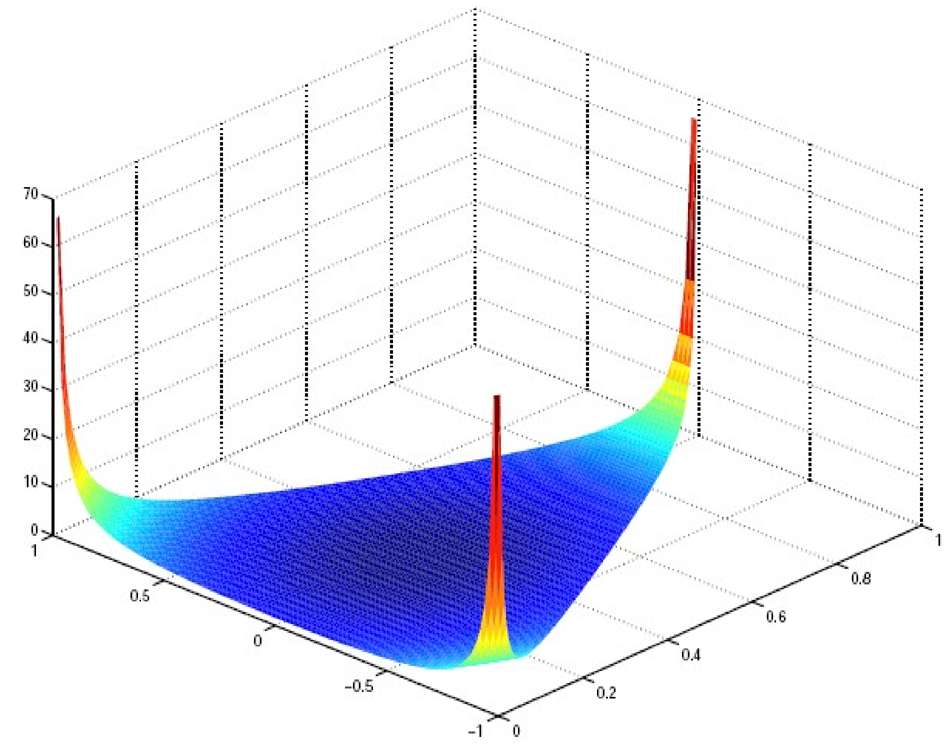
$$\begin{aligned} b_{l_1 l_2 l_3} \quad = \quad & \left(\frac{2}{\pi}\right)^3 \int x^2 dx \left(\int k^2 dk X(k) \Delta_{l_1}(k) j_{l_1}(kx)\right) \\ & \left(\int k^2 dk Y(k) \Delta_{l_2}(k) j_{l_2}(kx)\right) \left(\int k^2 dk Z(k) \Delta_{l_3}(k) j_{l_3}(kx)\right) \end{aligned}$$

- There are currently two approximations that are used to cover all models
  - The local model where the non-Gaussianity is created on superhorizon scales
  - The equilateral model where the non-Gaussianity is created at horizon crossing

Current Approach

- Local model
  - Found by taking a Taylor expansion around a Gaussian for the gravitational potential perturbation
  - Examples include the curvaton model and multiple field inflation models

Komatsu, Spergel astro-ph/00050036



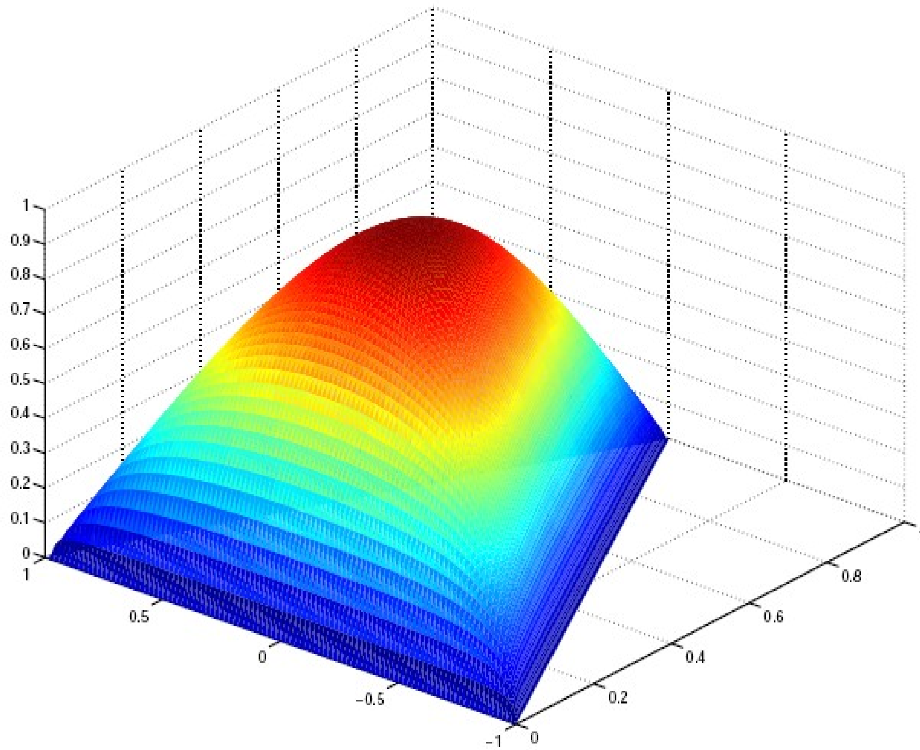
$$\Psi(x) \approx \Psi_L(x) + f_{NL}\left(\Psi_L^2(x) - \langle\Psi_L^2(x)\rangle\right)$$

$$F(k_1,k_2,k_3) = 2f_{NL}\left(P^\Psi(k_1)P^\Psi(k_2) + P^\Psi(k_2)P^\Psi(k_3) + P^\Psi(k_3)P^\Psi(k_1)\right)$$

Current Approach

- Equilateral model
  - This is an approximation found by hand for models where we have a non-minimal Lagrangian and the correlation is created by higher derivative operators
  - Examples include the DBI model and ghost inflation

Creminelli, Nicolis, Senatore, Tegmark, Zaldarriaga astro-ph/0509029



$$s = \frac{1}{2}(k_1 + k_2 + k_3)$$

$$F(k_1, k_2, k_3) = A\frac{(s-k_1)(s-k_2)(s-k_3)}{k_1^3 k_2^3 k_3^3}$$

Current Approach

- **Pros**
  - 4D problem reduces to 2D so much faster to calculate
  - Provides nice way to calculate estimators (see later)
- **Cons**
  - Loss of generality: primordial bispectrum must be well approximated by a separable analytic formula

New Approach

We know that the primordial bispectrum must exhibit near scale invariance

$$F(k,k,k) \propto k^{-6}$$

So perhaps if we reparametrised the primordial bispectrum so that we could separate out the scale part we might make some progress

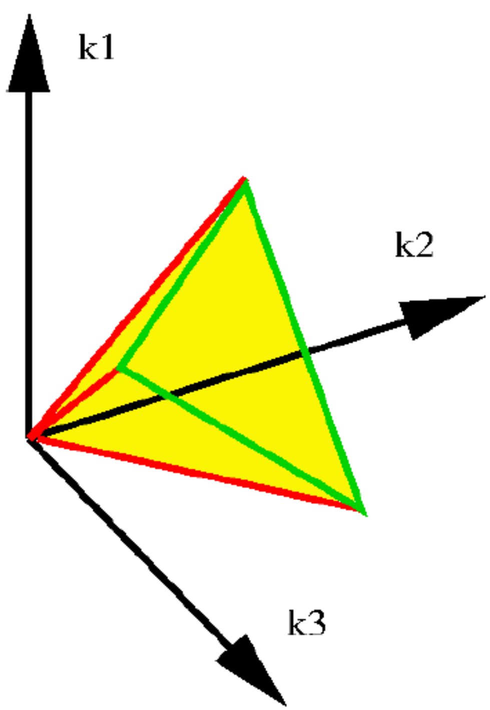
New Approach

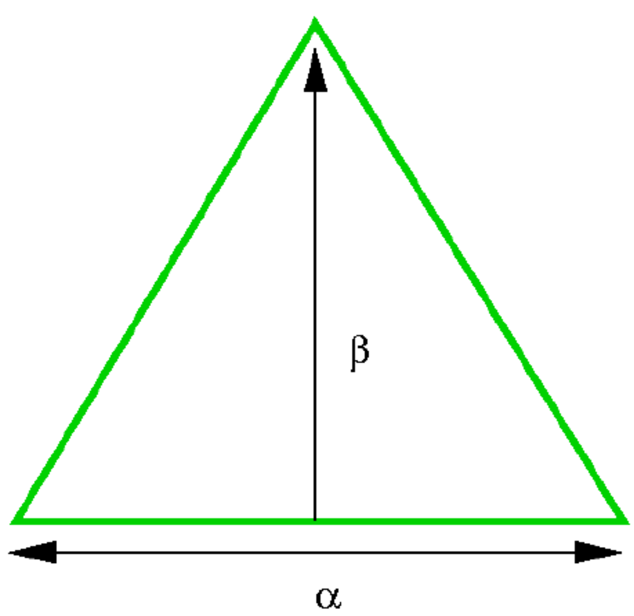
We re-parametrise the region in k-space into a distance along the tetrahedron and our point on the slice perpendicular

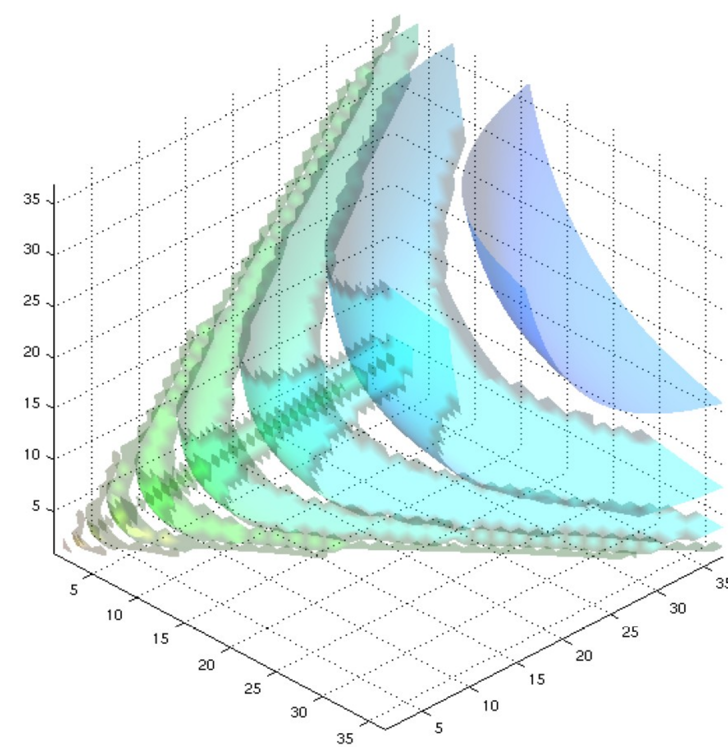
$$k_1 = \; ka \; = k\left(1-\beta\right) \qquad\qquad 0 \leq k < \infty$$

$$k_2 = \; kb \; = \frac{1}{2}k\left(1+\alpha+\beta\right) \qquad\qquad 0 \leq \beta \leq 1$$

$$k_3 = \; kc \; = \frac{1}{2}k\left(1-\alpha+\beta\right), \qquad\qquad -(1-\beta) \leq \alpha \leq 1-\beta$$







New Approach

- We assume that the shape function splits into a scale part and a scale independent part

$$F(k_1, k_2, k_3) \approx \frac{k^n}{k^6} F(\alpha, \beta)$$

- The integral separates into three parts integrated over the triangular slice

$$\left(\frac{2}{\pi}\right)^3 \int d\alpha d\beta \, F^{SI}(\alpha, \beta) \, I^T(\alpha, \beta) \, I^G(\alpha, \beta)$$

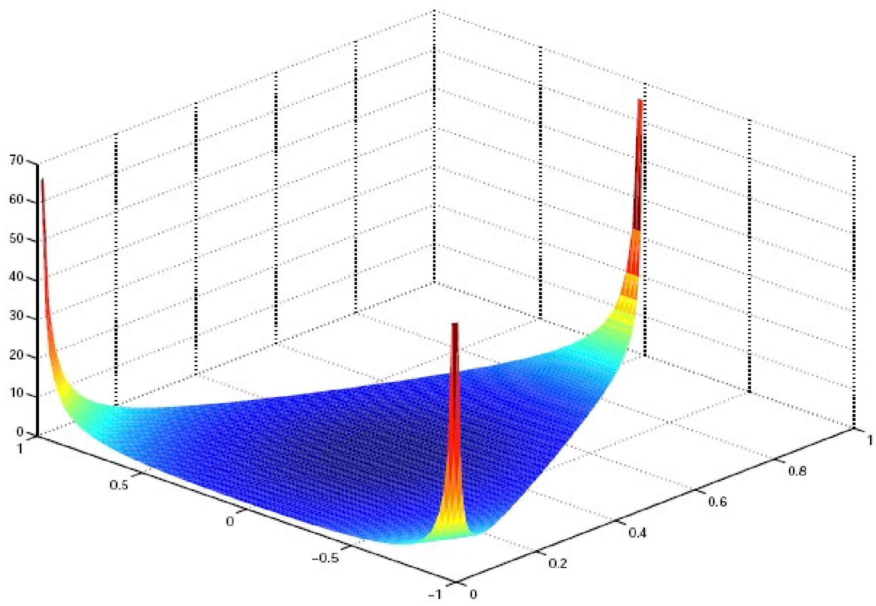
- All the highly oscillatory functions are trapped in one dimensional integrals

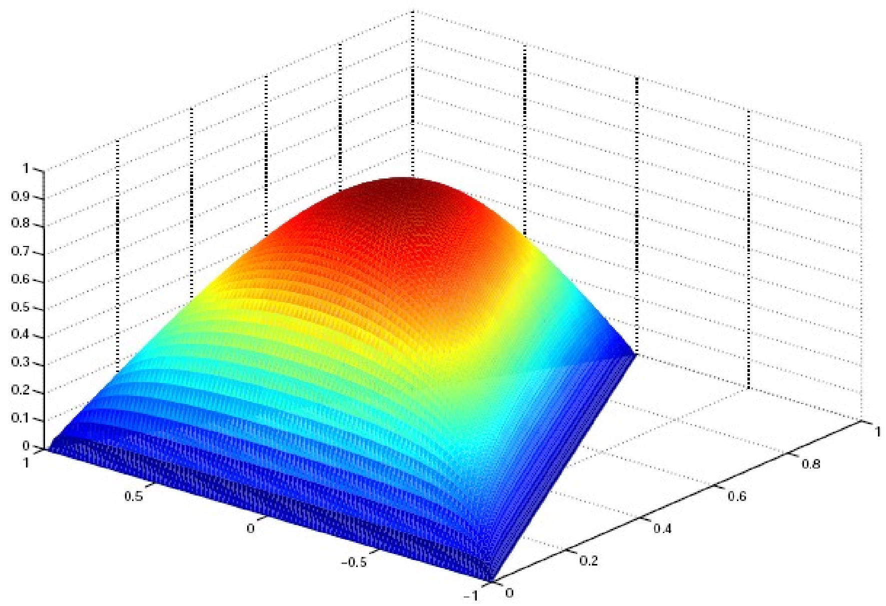
$$\begin{aligned} I^T(\alpha, \beta) &\equiv \int \Delta_{l_1}(ak) \, \Delta_{l_2}(bk) \, \Delta_{l_3}(ck) \, k^n \frac{dk}{k} \\ I^G(\alpha, \beta) &\equiv \int j_{l_1}(ax) \, j_{l_2}(bx) \, j_{l_3}(cx) \, x^2 dx \\ F^{SI}(\alpha, \beta) &\equiv (abc)^2 \, F(\alpha, \beta), \end{aligned}$$

New Approach

$F^{SI}(\alpha, \beta)$ plots

- Local model
- Equilateral model





New Approach

- **Pros**
  - 4D reduced to 3D
  - All highly oscillatory behaviour confined to the two 1D integrals
  - We expect all realistic primordial bispectrums to be well approximated by the scaling assumption so the solution is general
  - Can handle numerical primordial bispectrums
- **Cons**
  - Not as fast to calculate as the fully separable case
  - Difficulties with measurement (see later)

Numerical Work

First we will derive a analytic solution to test against.

- Local model bispectrum $$F^{SI}(k_1, k_2, k_3) = \frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_3 k_1} + \frac{k_3^2}{k_1 k_2}.$$

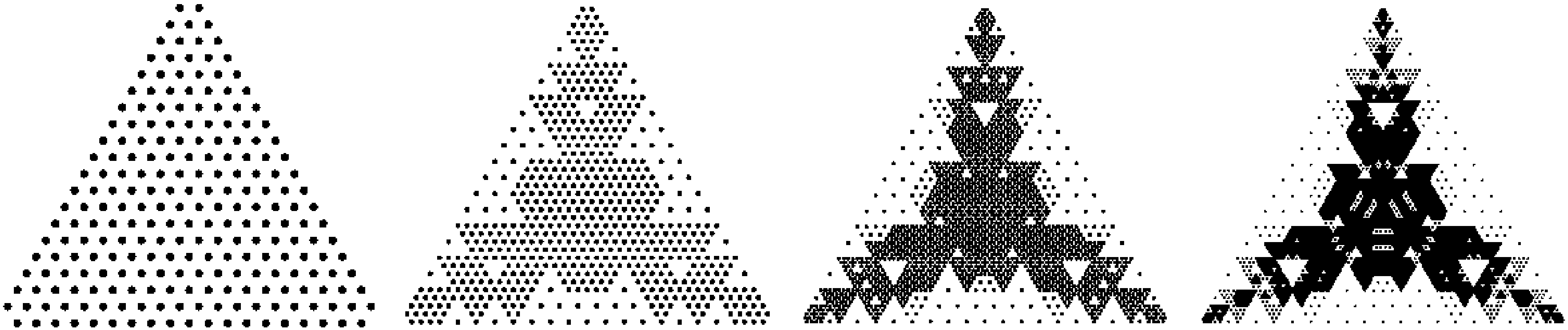
- Large angle approximation $$\Delta_l(k) = \frac{1}{3} j_l(\Delta\eta\, k)$$

If we substitute and integrate we get an analytic expression for the reduced bispectrum

$$\left(\frac{1}{27\pi^2}\right)\left(\frac{l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1)}{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)}\right)$$

Numerical Work

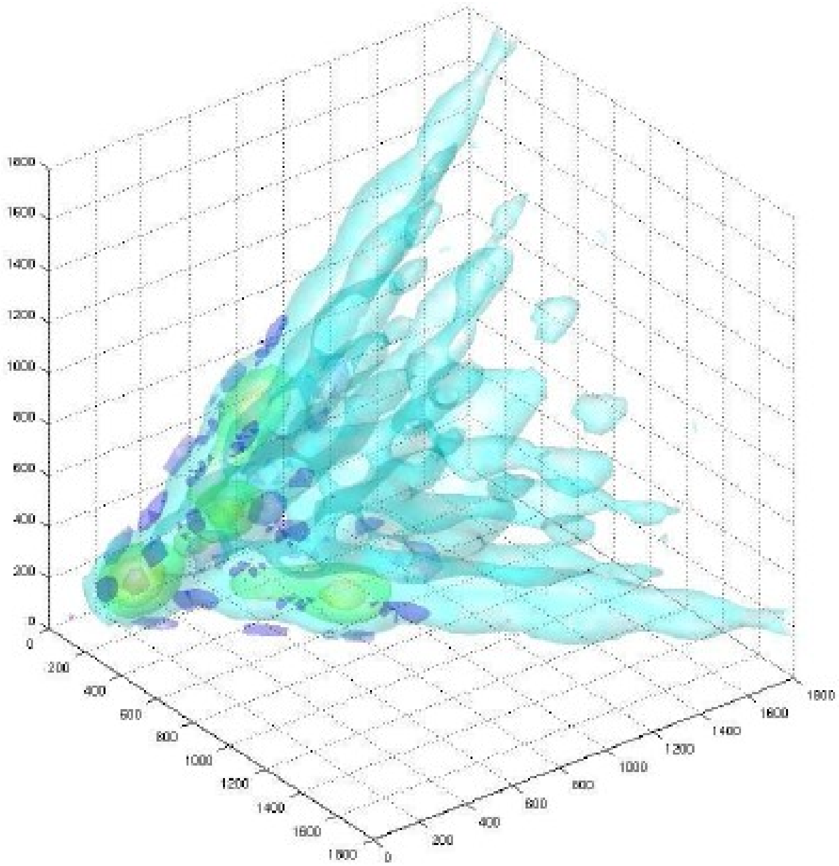
- The 1D integrals are computed using the trapezoidal rule from tabulated functions
- The 2D integral on the triangle is computed using a grid refinement algorithm



Results

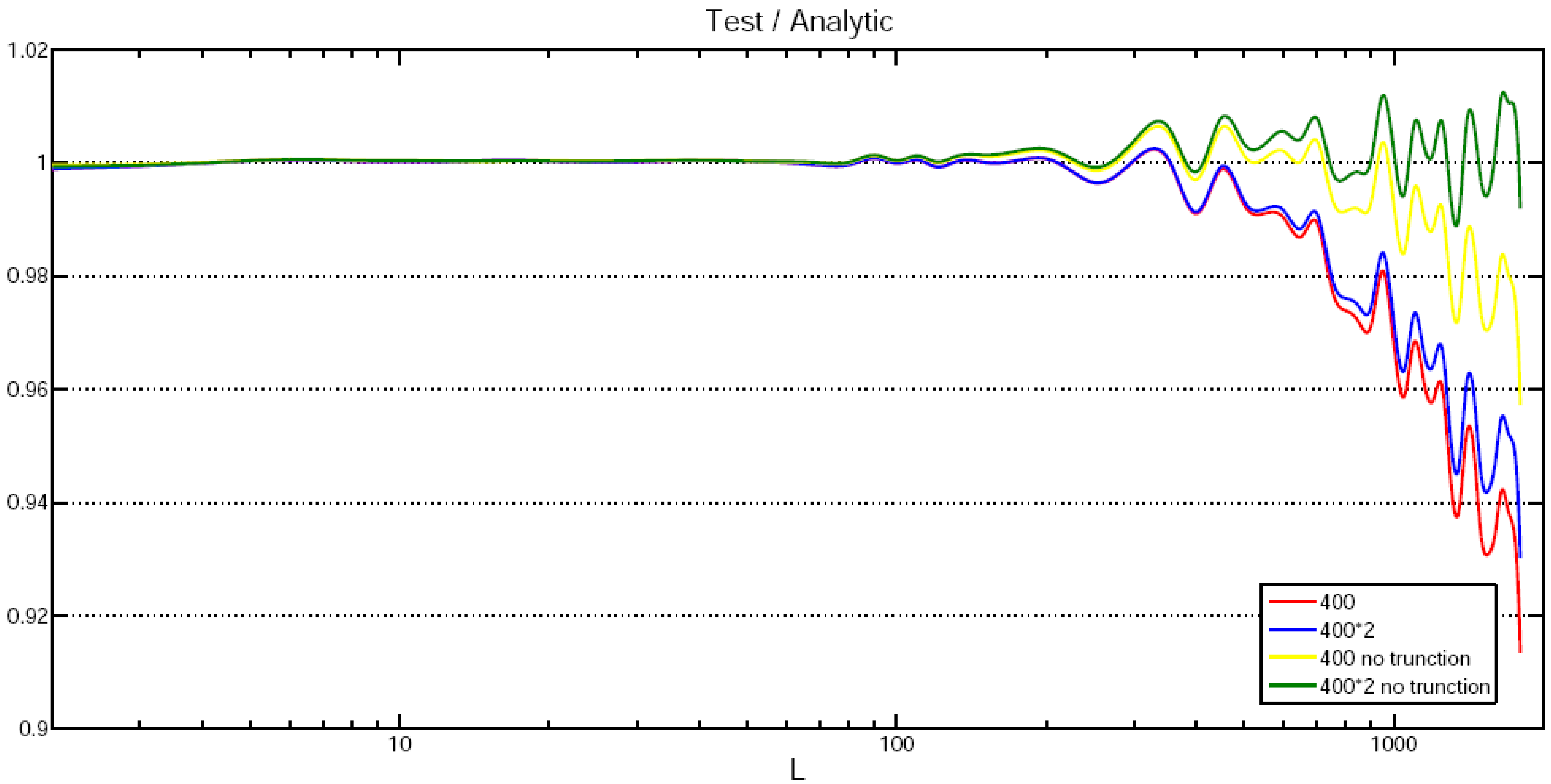
- To plot results we split l-space in the same way we did for k-space

$$
\begin{aligned}
l_1 &= \frac{3}{2} l \left(1 - \gamma\right) \\
l_2 &= \frac{3}{4} l \left(1 + \delta + \gamma\right) \\
l_3 &= \frac{3}{4} l \left(1 - \delta + \gamma\right)
\end{aligned}
$$



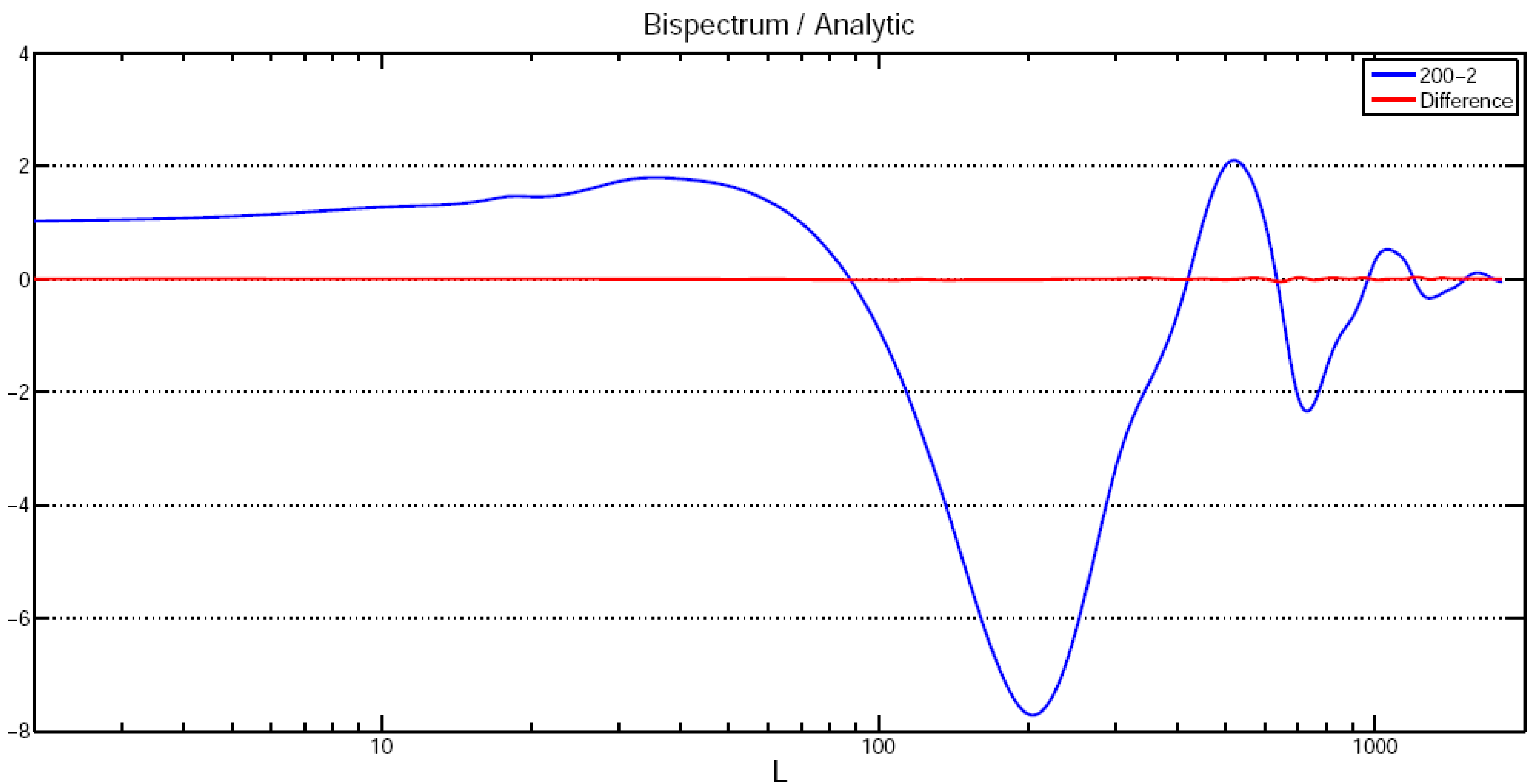
Results

- Effect of changing parameters for the 1D integrals for the local model



Results

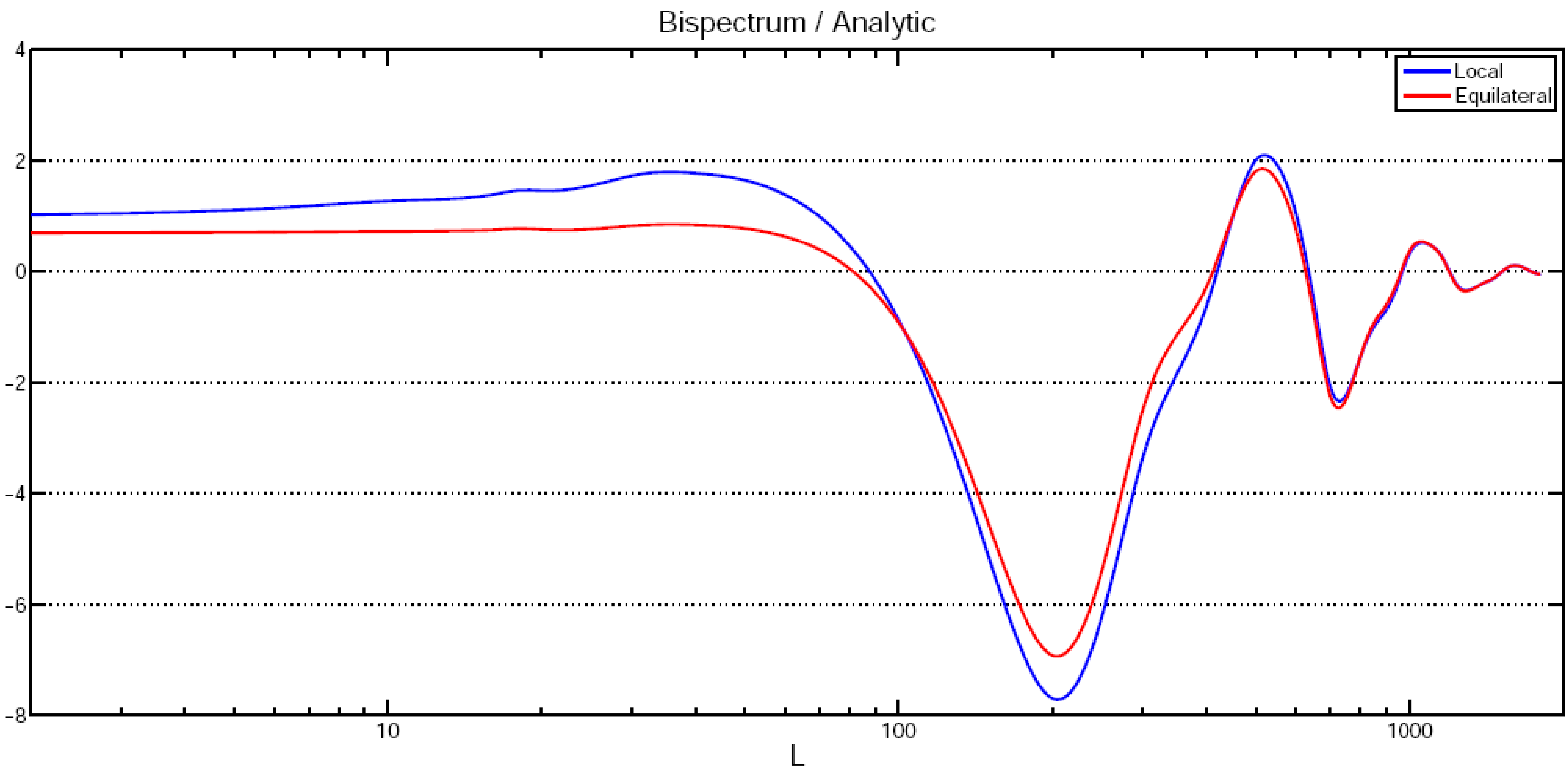
- Plot of the bispectrum for the local model with it's error



Results

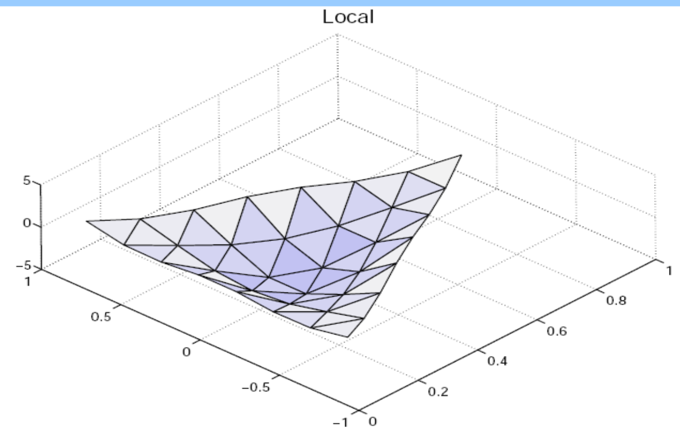
- Comparison of the local and equilateral models

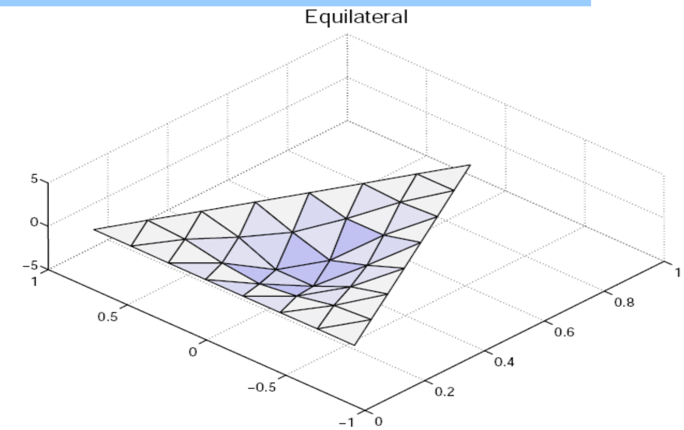
There are no large differences in the equal l bispectrum especially for large l. To see the differences between the models we must look at the transverse slices



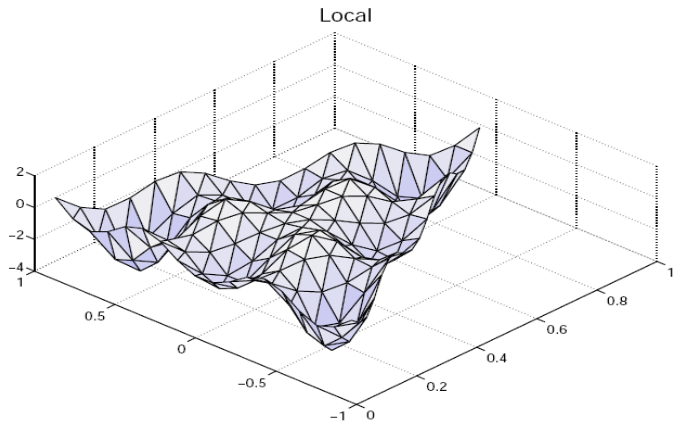
Results

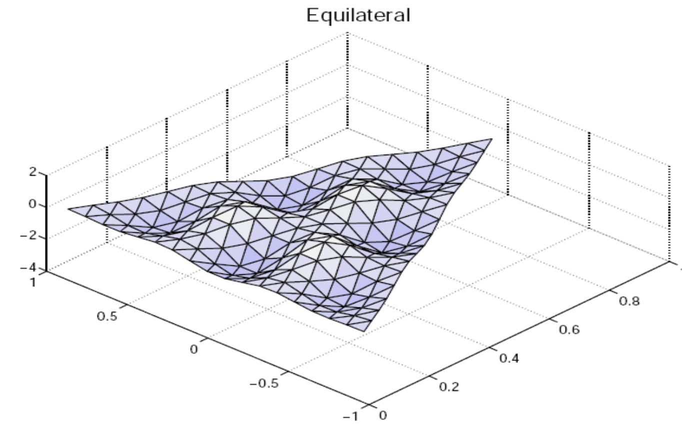
$3l = 850$



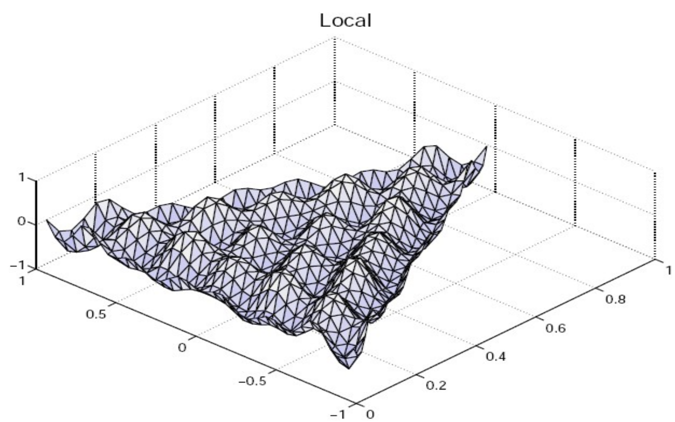


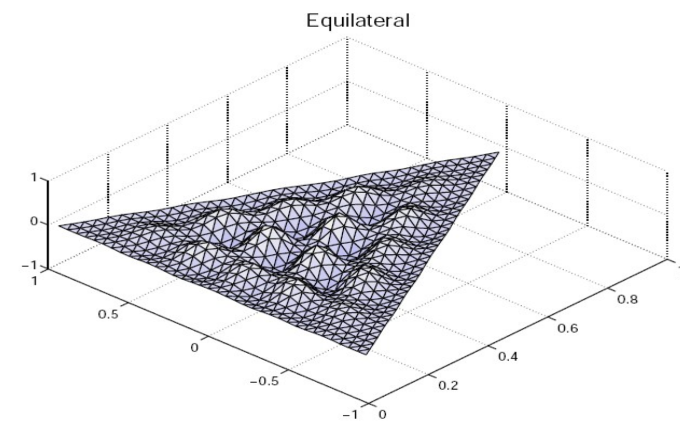
$3l = 1850$



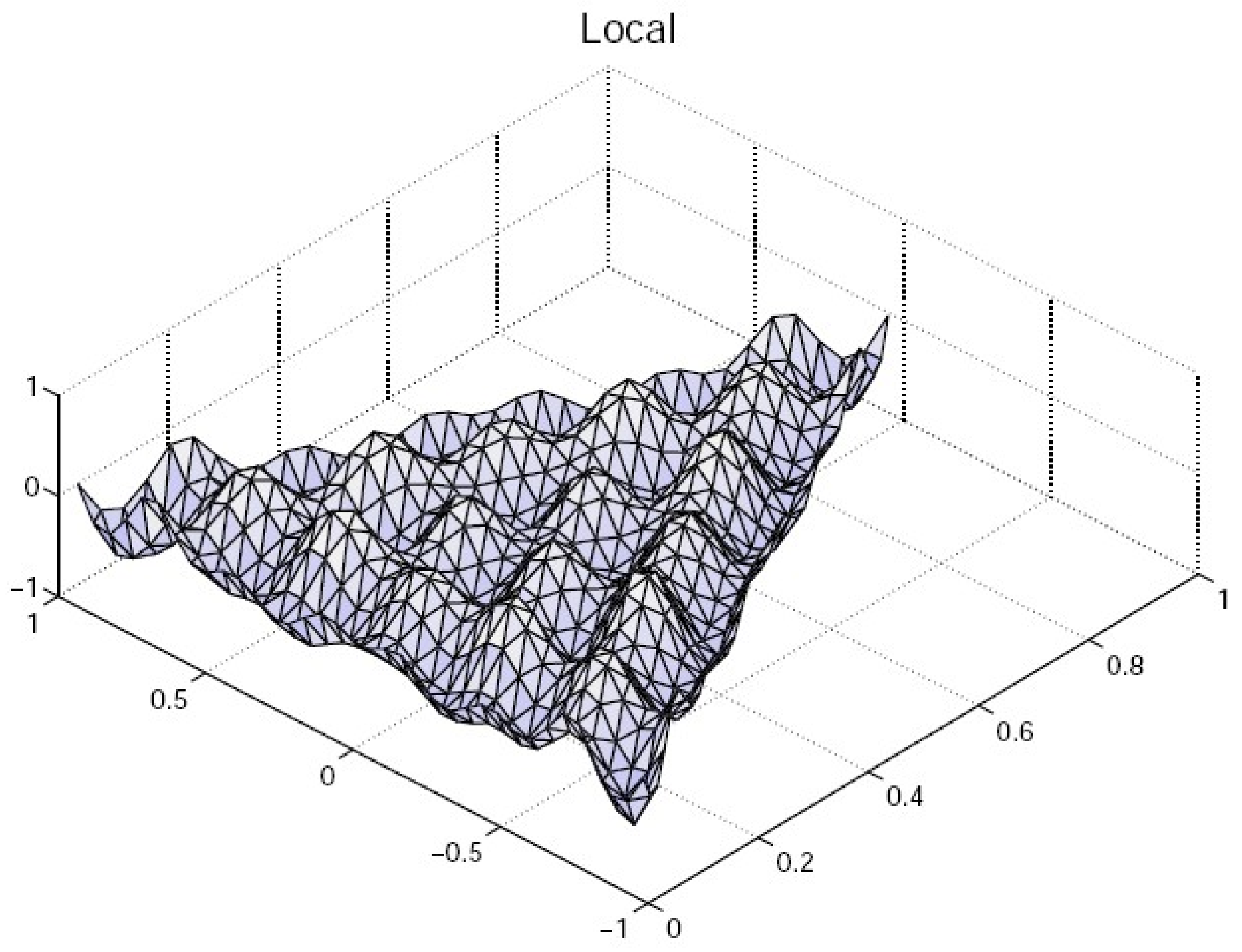


$3l = 3650$

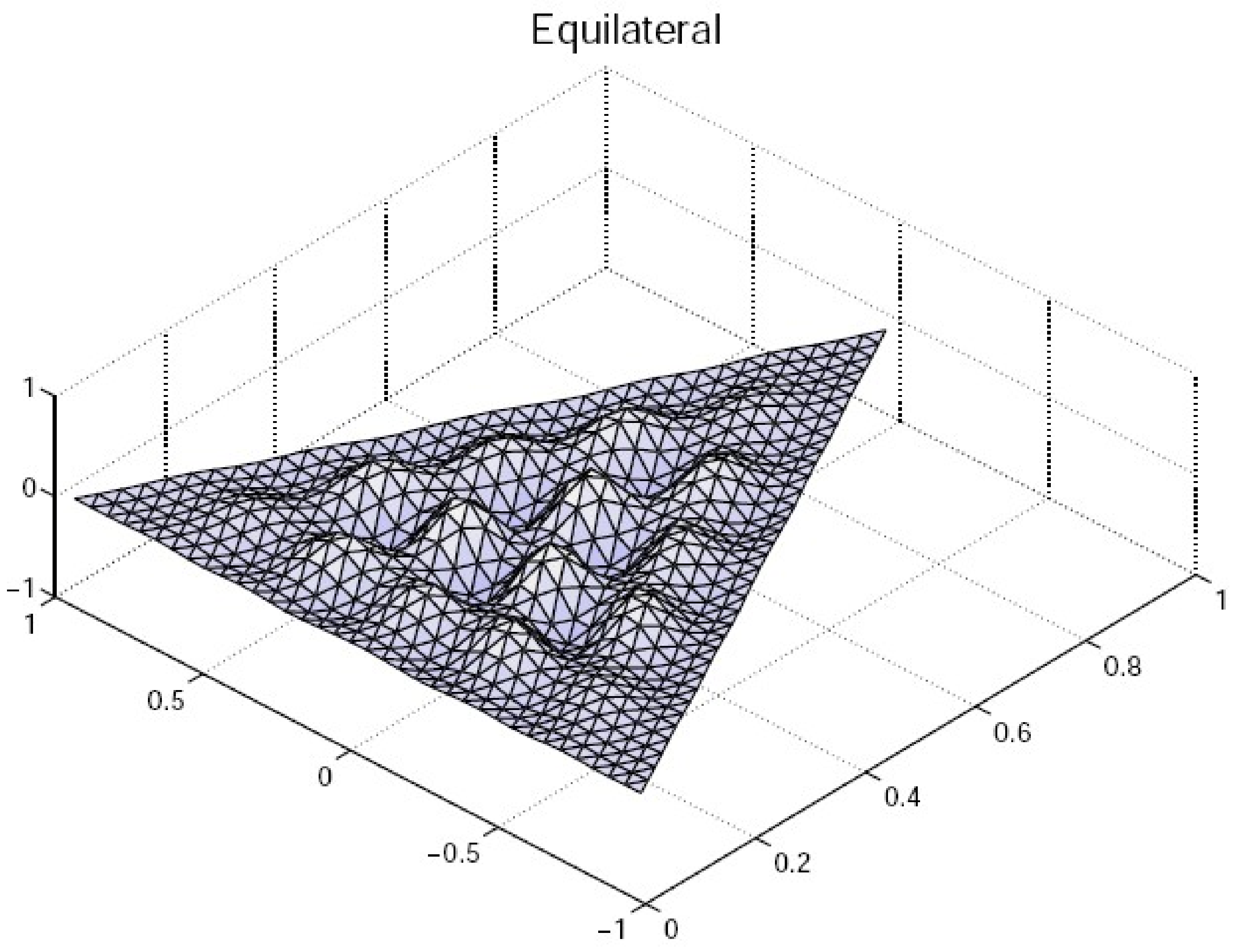




Results

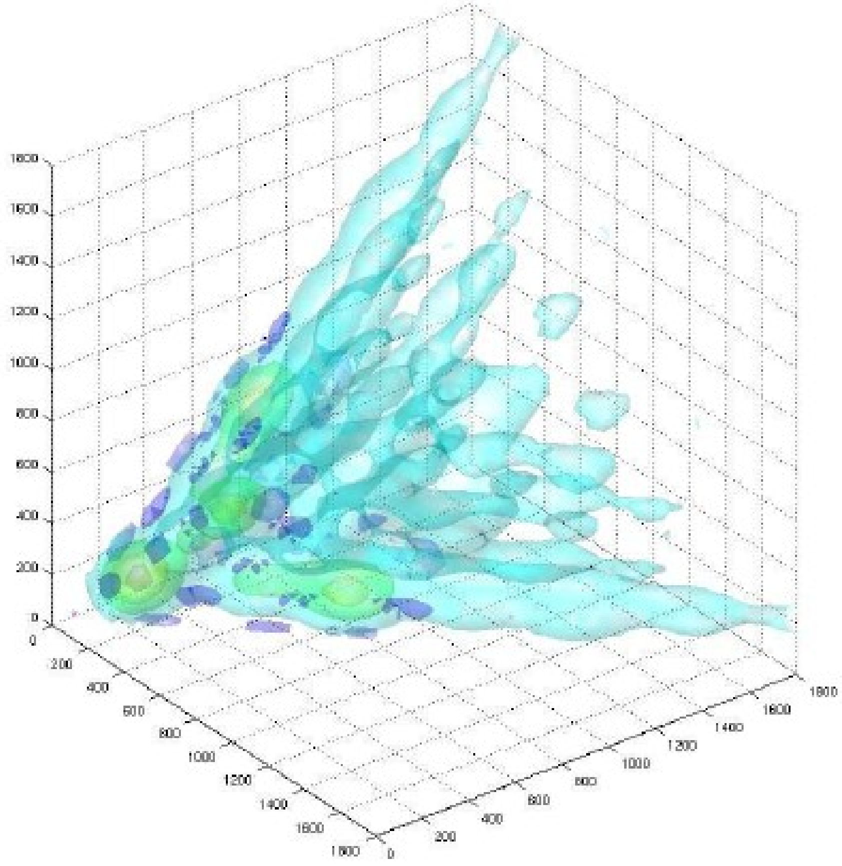


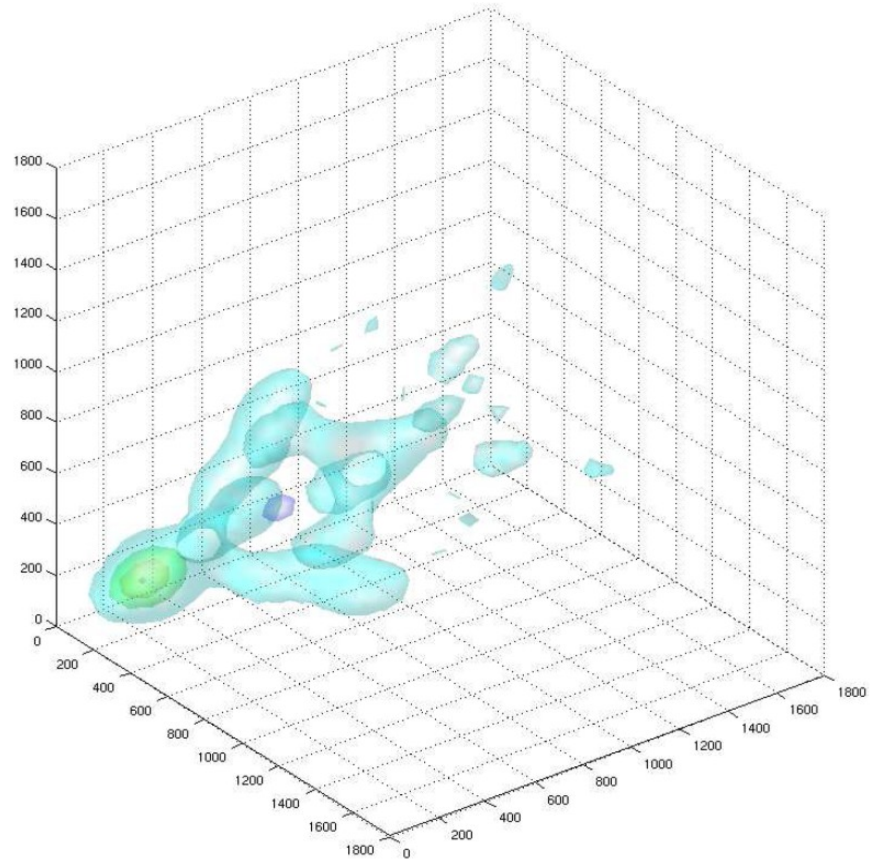
Results



Results

The differences are especially obvious in three dimensional plots of the reduced bispectrum where in the equilateral case the edges are heavily suppressed





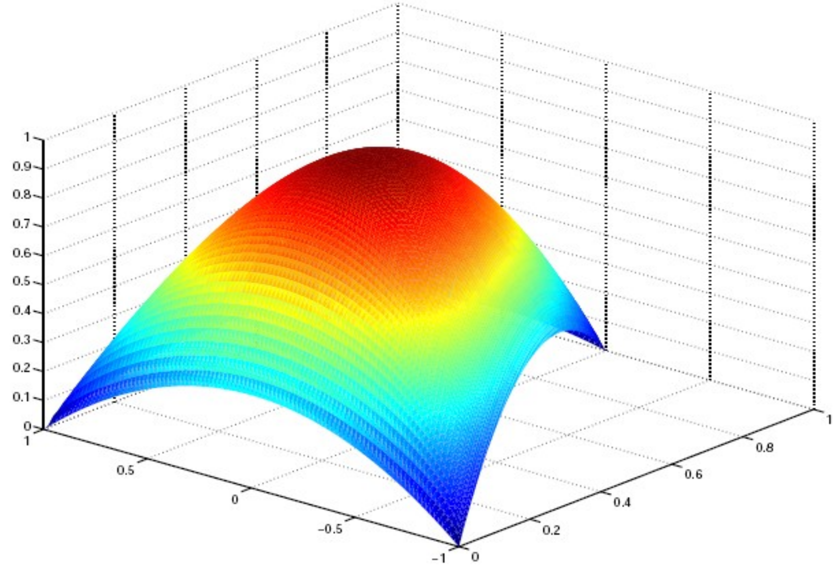
Results

DBI

What about a real non-separable model?

The equilateral model was created to approximate the DBI model but now we can calculate it in full

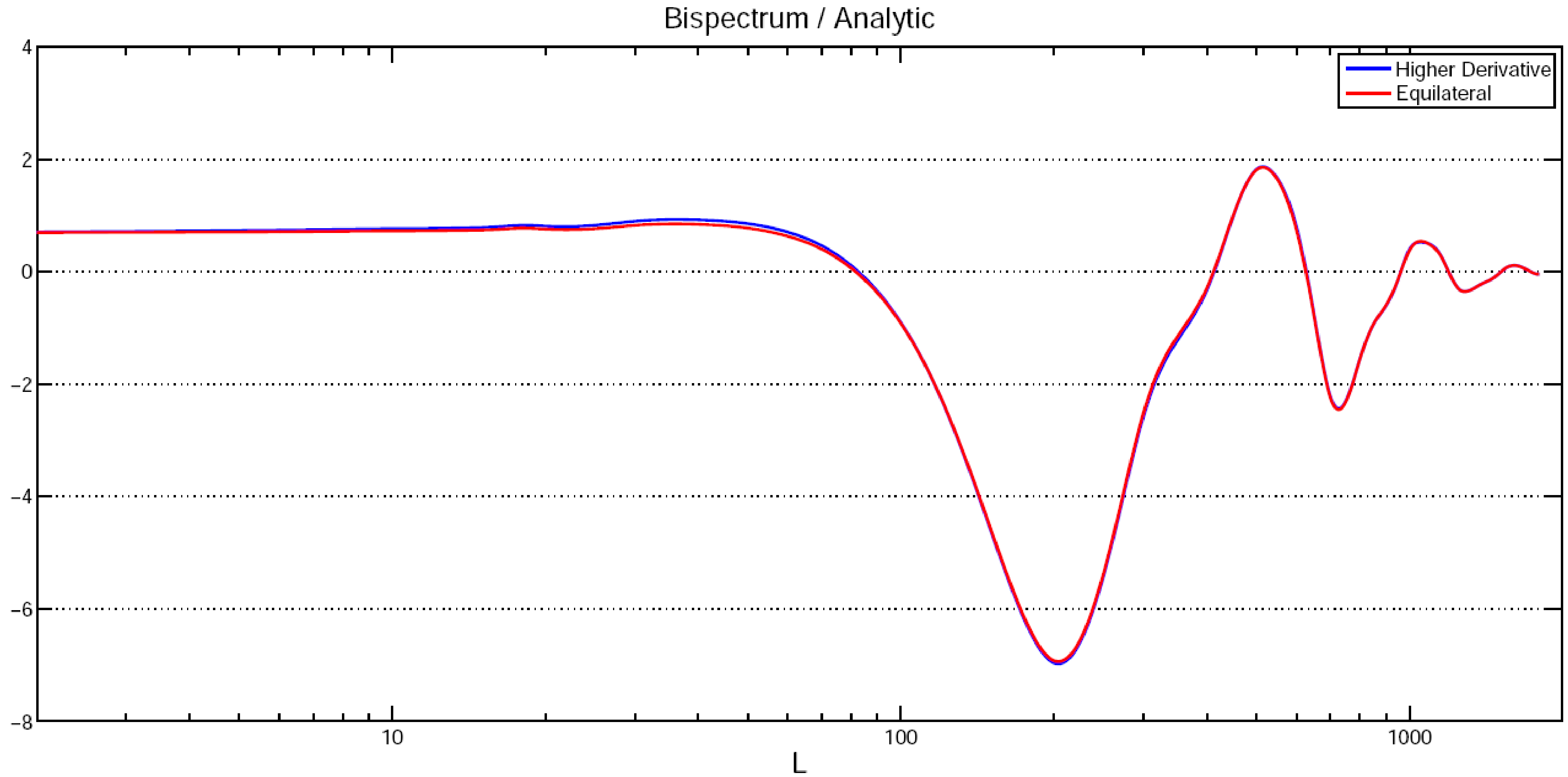
$$F^{SI}(k_1,k_2,k_3) = \frac{1}{k_1k_2k_3(k_1+k_2+k_3)^2}\left(\sum_i k_i^5 + \sum_{i\neq j}(2k_i^4k_j - 3k_i^3k_j^2) + \sum_{i\neq j\neq l}(k_i^3k_jk_l - 4k_i^2k_j^2k_l)\right)$$



Results

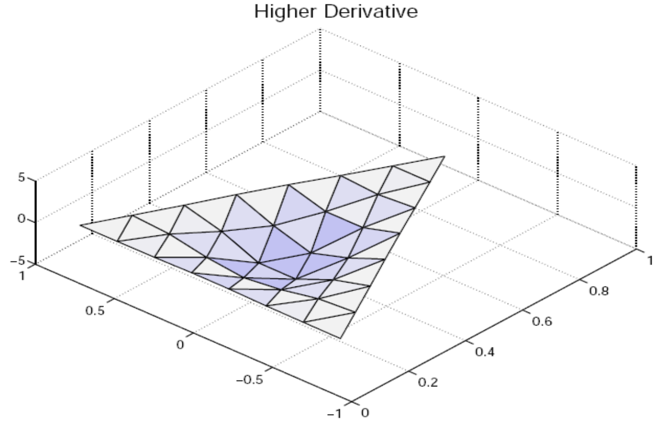
Comparison of equilateral and DBI bispectrums

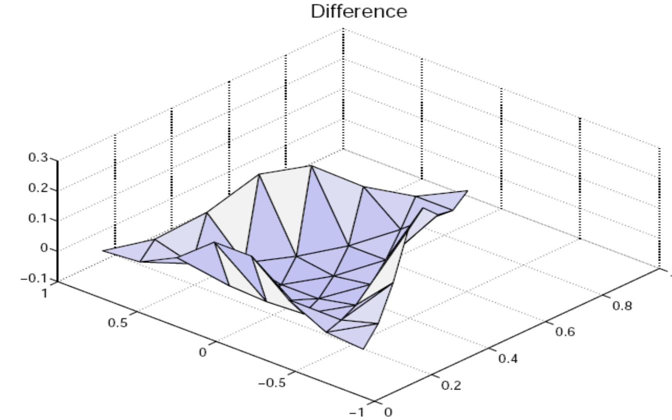
The equal l bispectrum for DBI is well approximated by the equilateral shape. What about the slices?



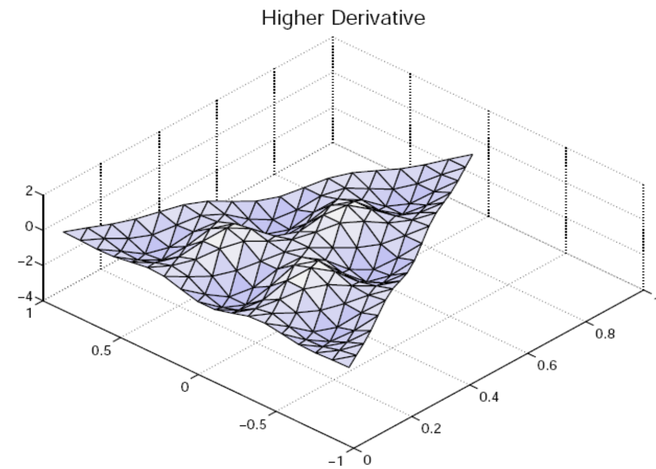
Results

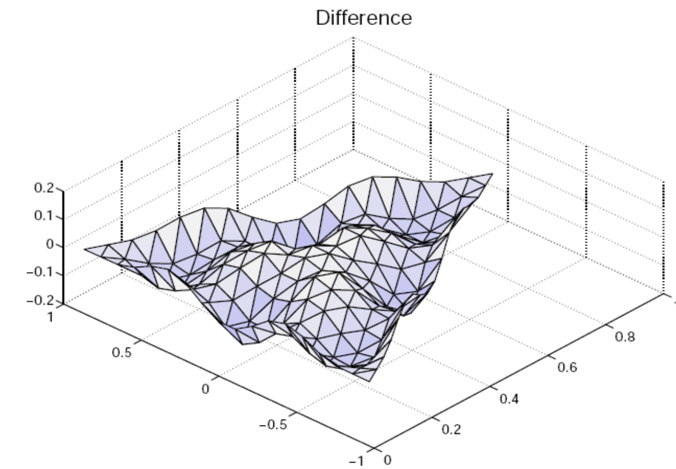
3l = 850



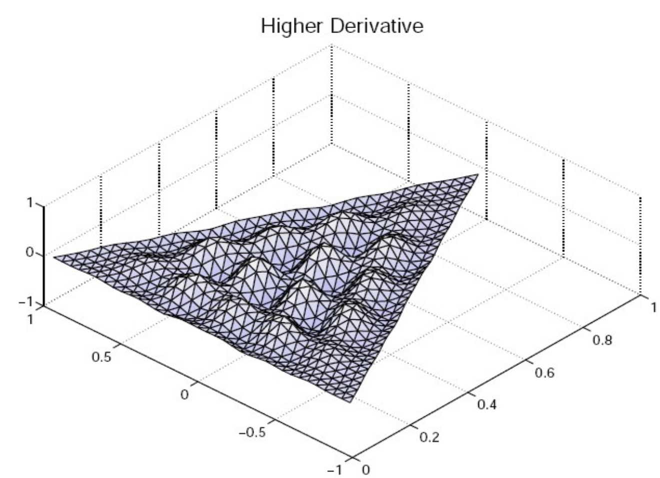


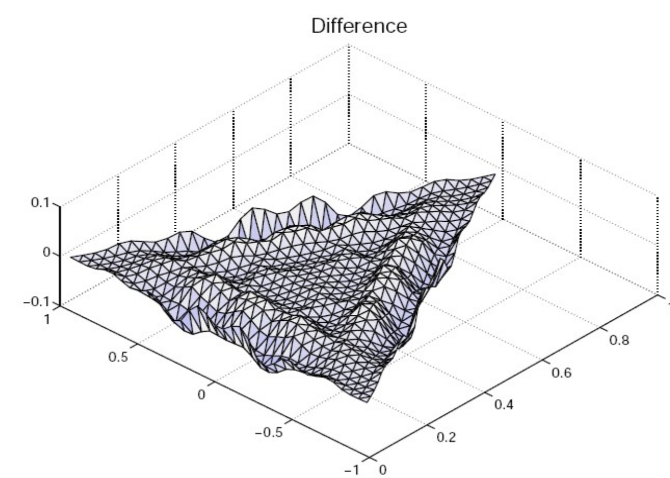
3l = 1850





3l = 3650

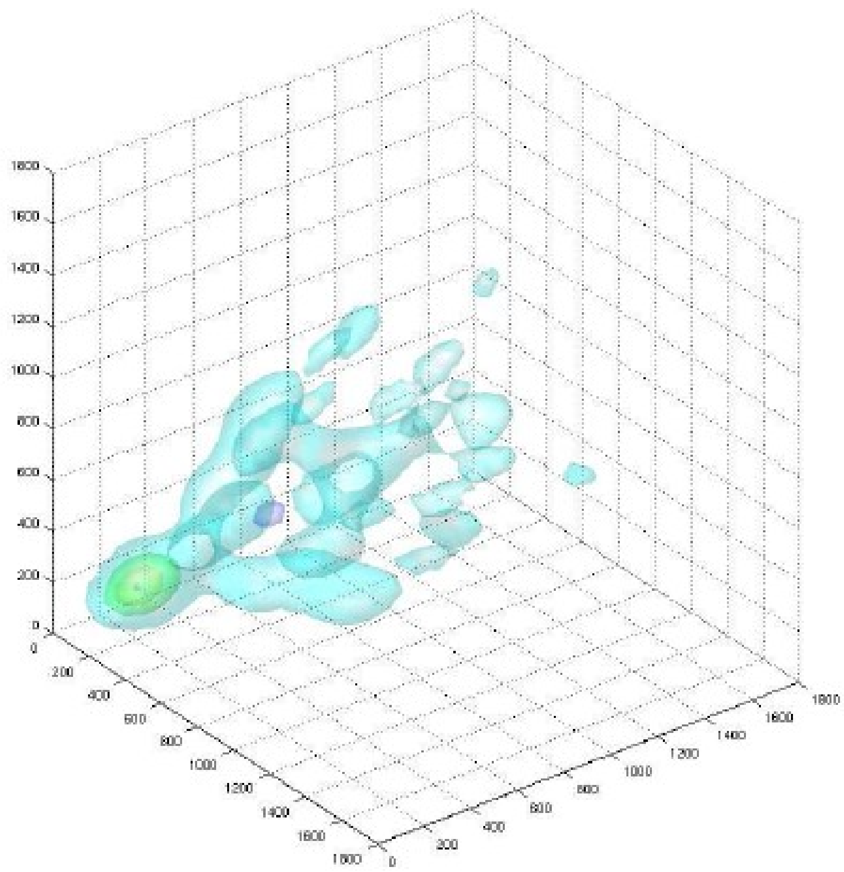




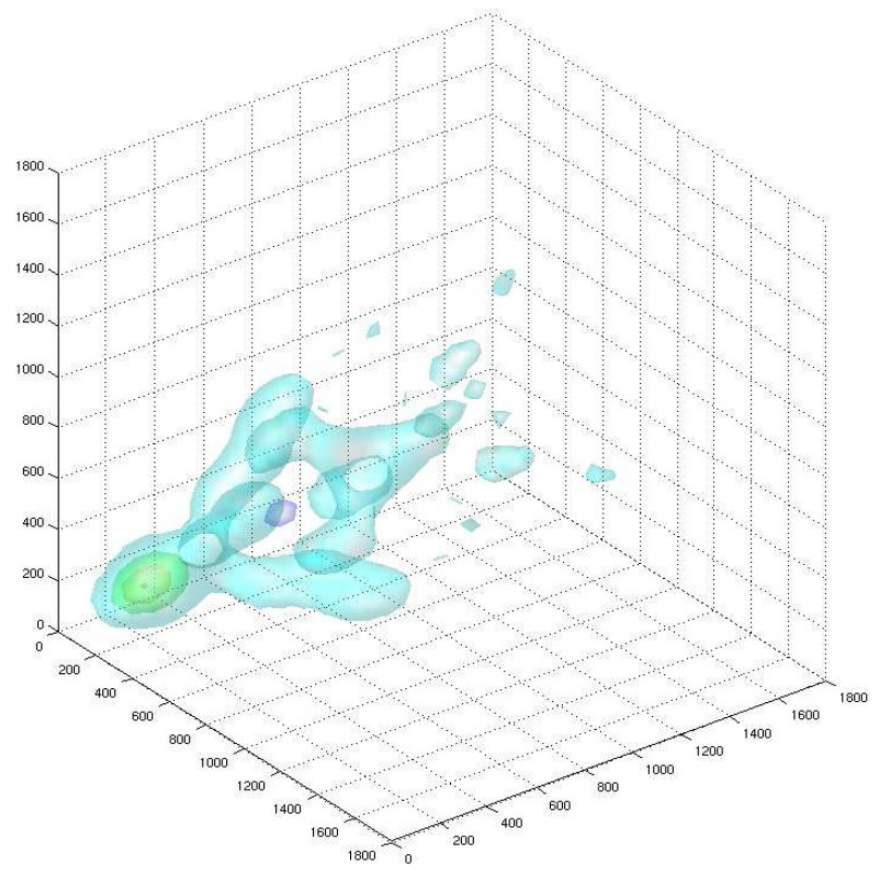
Results

3D comparison, in the equilateral case we have suppression at the sides.

DBI



Equilateral



Flat Sky

Calculating all the l combinations takes a long time so how can we speed it up? For large l we can use the flat sky approximation

- We decompose the temperature anisotropy into plane waves to produce a(l)

$$a(\mathbf{l}) = \int d^2n \frac{\Delta T}{T}(\mathbf{x},\mathbf{n}) e^{i\mathbf{l}\cdot\mathbf{n}}$$

- The temperature perturbation can be represented in terms of the primordial gravitational potential perturbation times the line of sight integral of the source function

$$\begin{aligned}\frac{\Delta T}{T}(\mathbf{x},\mathbf{n}) &= \int \frac{d^3k}{(2\pi)^3} \frac{\Delta T}{T}(\mathbf{k},\mathbf{n}) \\ &= \int \frac{d^3k}{(2\pi)^3} \Psi(k) \int_0^{\tau_0} d\tau S(k,\tau) e^{-ik\mu(\tau_0-\tau)}\end{aligned}$$

- Substituting gives an expression for a(l)

$$a(\mathbf{l}) = \int \frac{d^3k}{(2\pi)} \Psi(k) \int_0^{\tau_0} d\tau S(k,\tau) e^{-ik^z(\tau_0-\tau)} \delta^2(\mathbf{k}^{\|}(\tau_0-\tau)-\mathbf{l})$$

Babich, Creminelli, Zaldarriaga astro-ph/0405356

Flat Sky

- We form the three point correlator for the a(l)'s

$$\begin{aligned}\langle a(\mathbf{l}_1)a(\mathbf{l}_2)a(\mathbf{l}_3)\rangle &= (\tau_0-\tau_R)^2\delta^2(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3)\int d\tau_1 d\tau_2 d\tau_3 dk_1^z dk_2^z dk_3^z \\ &\quad \delta(k_1^z+k_2^z+k_3^z)F(k_1',k_2',k_3')S(k_1',\tau_1)S(k_2',\tau_2)S(k_3',\tau_3) \\ &\quad e^{-ik_1^z(\tau_0-\tau_1)}e^{-ik_2^z(\tau_0-\tau_2)}e^{-ik_3^z(\tau_0-\tau_3)}\end{aligned}$$

- This is related to the flat reduced bispectrum times a delta function and for large l the two reduced bispectrums are the same

$$\langle a(\mathbf{l}_1)a(\mathbf{l}_2)a(\mathbf{l}_3)\rangle = (2\pi)^2\delta^2(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3)b_{l_1l_2l_3}$$

$$b_{l_1l_2l_3} \approx b^{flat}_{l_1l_2l_3}$$

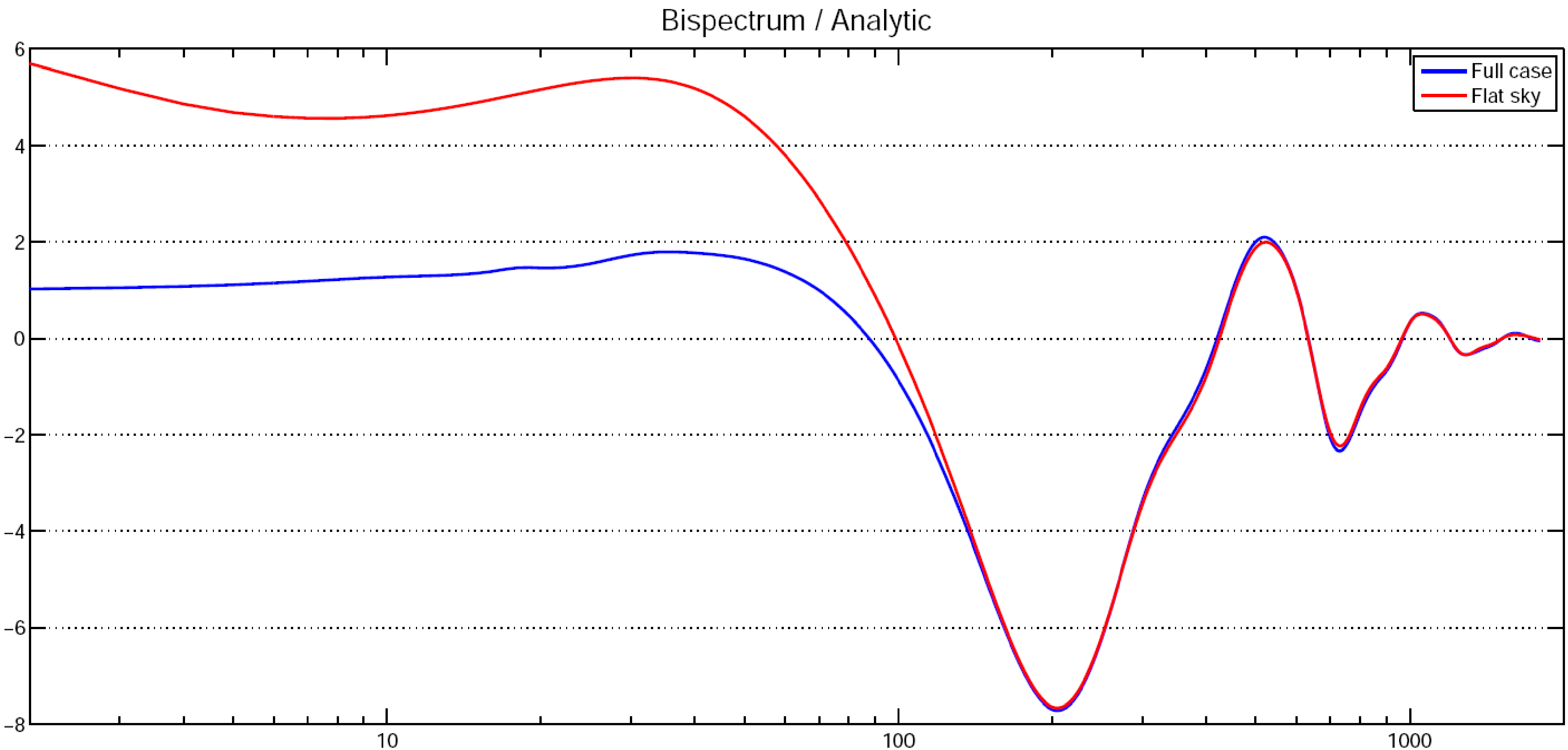
- So defining a new transfer function we have a formula for the reduced bispectrum for large l which only involves 2D-integration

$$\Delta(l,k^z) = \int \frac{d\tau}{(\tau_0-\tau)^2}S(\sqrt{(k^z)^2+l^2/(\tau_0-\tau)^2},\tau)e^{ik^z\tau}$$

$$b_{l_1l_2l_3} = \frac{(\tau_0-\tau_R)^2}{(2\pi)^2}\int dk_1^z dk_2^z dk_3^z\delta(k_1^z+k_2^z+k_3^z)F(k_1',k_2',k_3')\Delta(l_1,k_1^z)\Delta(l_2,k_2^z)\Delta(l_3,k_3^z)$$

Flat Sky

The flat sky approximation allows bispectrum points to be accurately calculated up to 100 times faster when all three l's are >=150



Measurement

It is unlikely that we will be able to measure the bispectrum directly so we must use estimators to establish limits on the magnitude of non-Gaussianity for different models.

$$S = \frac{1}{N} \sum_{l_i m_i} \frac{\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}}{C_{l_1} C_{l_2} C_{l_3}}$$

$$N = \sum_{l_i m_i} \frac{\left(\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3}\right)^2}{C_{l_1} C_{l_2} C_{l_3}}$$

Problem: For a maximum l of 335 (as used in WMAP's analysis) there are 80 billion independent Wigner 3j-symbols we need to calculate for the Gaunt integral and they are difficult to calculate quickly as there are no simple analytic formula

$$\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Measurement

Current approach:

For the separable case we can replace Gaunt integral with an integral over three spherical harmonics then absorb these into the integral for each l

$$F(k_1 k_2 k_3) = \prod_i^3 F_i(k_i)$$

$$b_{l_1 l_2 l_3} = \int x^2 dx \prod_i^3 \frac{2}{\pi} \int k^2 dk F_i(k) \Delta_{l_i}(k) j_{l_i}(kx)$$

$$S = \frac{1}{N} \int d^2 n \int x^2 dx \prod_i^3 \sum_{l_i m_i} \frac{2}{\pi} \int k^2 dk F_i(k) \Delta_{l_i}(k) j_{l_i}(kx) \frac{a_{l_i m_i} Y_{l_i m_i}(\mathbf{n})}{C_{l_i}}$$

Measurement

- The reduced bispectrum is smooth so if we could decompose it into the product of basis functions, each of a single l, then we could absorb the spherical harmonics as before.

If we try expanding the reduced bispectrum over the analytic solution which is always of similar order and slowly oscillating then the shifted Legendre polynomials should be suitable

$$\left(\frac{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)}{l_1(l_1+1)+l_2(l_2+1)+l_3(l_3+1)}\right) b_{l_1 l_2 l_3} = \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} X_\alpha(l_1) X_\beta(l_2) X_\gamma(l_3)$$

To do this we define a new polynomial X' which is the Legendre polynomial divided by l(l+1)

$$X_\alpha(l) = P_\alpha\left(\frac{2l - l_{max}}{l_{max}}\right) \qquad X'_\alpha(l) = \frac{X_\alpha(l)}{l(l+1)}$$

$$b_{l_1 l_2 l_3} = \frac{1}{3} \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} \left(X'_\alpha(l_1) X'_\beta(l_2) X_\gamma(l_3) + 2 \text{ permutations} \right)$$

(Smith, Zaldarriaga astro-ph/0612571)

Measurement

With these definition then we can use the orthogonality condition for the Legendre polynomials to find an expression for the expansion coefficient

$$\int_{-1}^{1} dx P_\alpha(x) P_\beta(x) = \frac{2\delta_{\alpha\beta}}{2\alpha+1} \quad \Rightarrow \quad \int_0^{l_{max}} \frac{dl}{l_{max}} X_\alpha(l) X_\beta(l) = \frac{\delta_{\alpha\beta}}{2\alpha+1}.$$

$$a_{\alpha\beta\gamma} = (2\alpha+1)(2\beta+1)(2\gamma+1) \int \frac{dl_1 dl_2 dl_3}{l_{max}^3} \left(\frac{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)}{l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1)} \right) b_{l_1 l_2 l_3} X_\alpha(l_1) X_\beta(l_2) X_\gamma(l_3).$$

It is now simple to form the estimator.

We define two filtered maps, one for each of the polynomials.

$$\bar{X}_\alpha(\hat{\mathbf{n}}) = \sum_{lm} X_\alpha(l) \frac{a_{lm}}{C_l} Y_{lm}(\hat{\mathbf{n}})$$

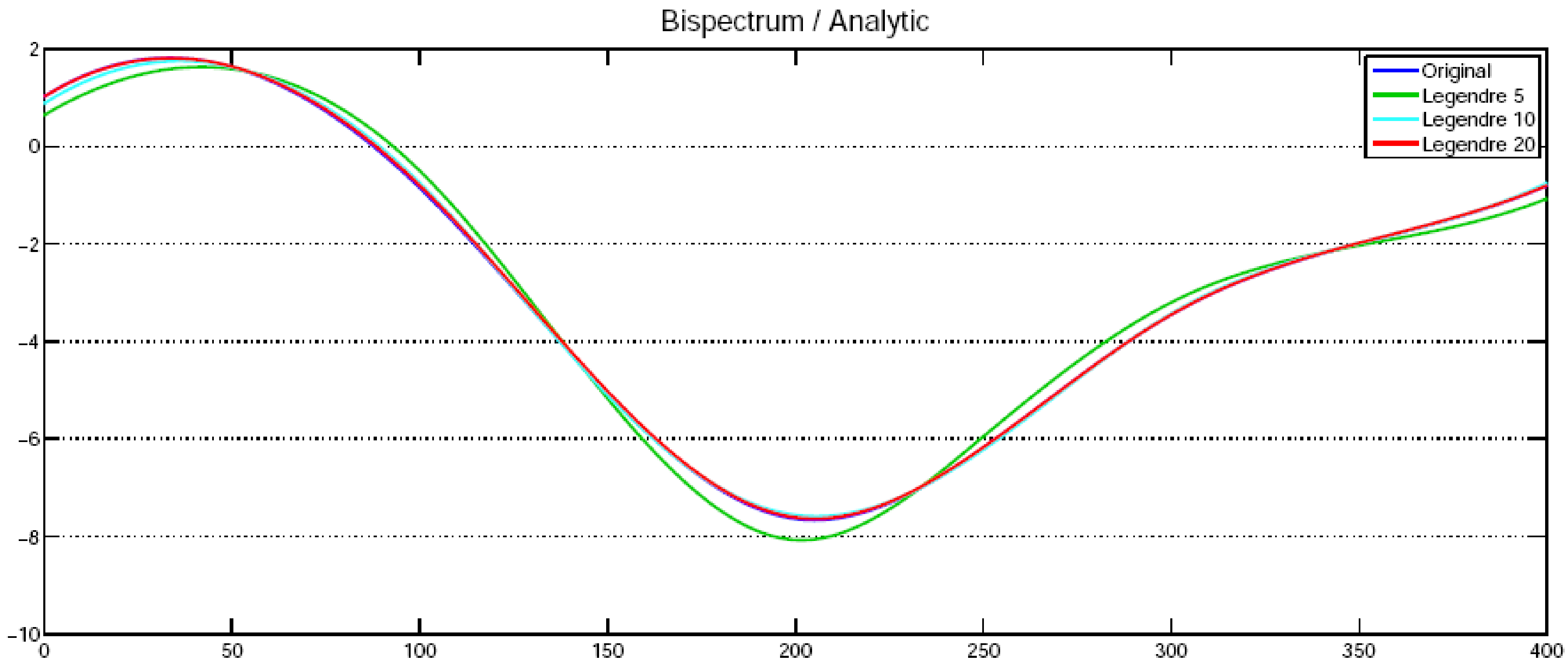
$$\bar{X}'_\alpha(\hat{\mathbf{n}}) = \sum_{lm} \frac{X_\alpha(l)}{l(l+1)} \frac{a_{lm}}{C_l} Y_{lm}(\hat{\mathbf{n}})$$

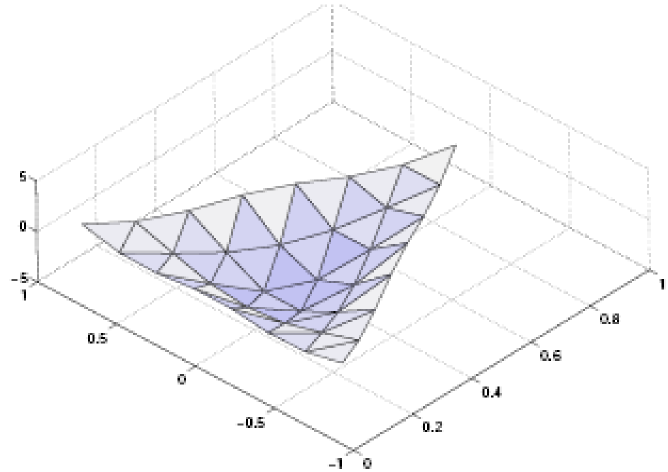
The estimator is then just the sum of the expansion coefficient times M, an integral over the maps

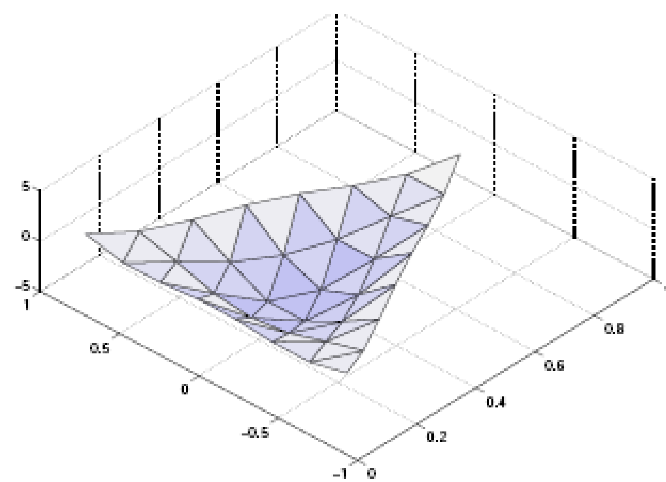
$$S = \frac{1}{N} \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} M_{\alpha\beta\gamma}.$$

$$M_{\alpha\beta\gamma} = \frac{1}{3} \int d\hat{\mathbf{n}} \left(\bar{X}'_\alpha(\hat{\mathbf{n}}) \bar{X}'_\beta(\hat{\mathbf{n}}) \bar{X}_\gamma(\hat{\mathbf{n}}) + 2 \text{ permutations} \right)$$

Measurement







Measurement

We have complete separation of theory and observation. Both $M_{\alpha\beta\gamma}$ and $a_{\alpha\beta\gamma}$ can be calculated completely independently. $M_{\alpha\beta\gamma}$ can be done once for each experiment and $a_{\alpha\beta\gamma}$ once per theory.

$$S = \frac{1}{N} \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} M_{\alpha\beta\gamma}$$

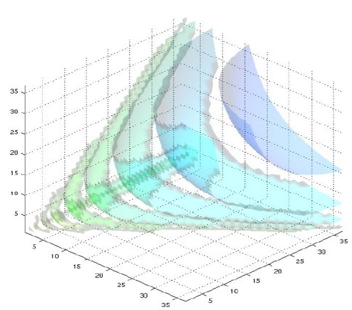
$$a_{\alpha\beta\gamma} = (2\alpha+1)(2\beta+1)(2\gamma+1) \int \frac{dl_1 dl_2 dl_3}{l_{max}^3} \left(\frac{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)}{l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1)} \right) b_{l_1 l_2 l_3} X_\alpha(l_1) X_\beta(l_2) X_\gamma(l_3).$$

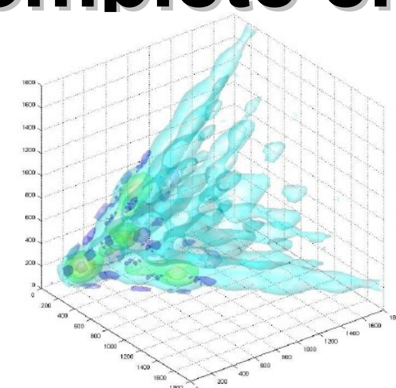
$$M_{\alpha\beta\gamma} = \frac{1}{3} \int d\hat{\mathrm{n}} \left(\bar{X}'_\alpha(\hat{\mathrm{n}}) \bar{X}'_\beta(\hat{\mathrm{n}}) \bar{X}_\gamma(\hat{\mathrm{n}}) + 2 \text{ permutations} \right)$$

Conclusions

- We can accurately calculate the bispectrum today from any primordial bispectrum
- With the flat sky approximation we can greatly speed calculation for $l_1, l_2, l_3 > 150$
- By decomposing the reduced bispectrum we can produce fast estimators and achieve a complete separation between theory and observation

We now have a complete end to end solution!





$$S = \frac{1}{N}\sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} M_{\alpha\beta\gamma}$$