Estimators For Extracting (Primordial) Non-Gaussianity

Eiichiro Komatsu
University of Texas, Austin
June 8, 2007
Why Study NG? (Why Care?!)

- Who said that CMB should be Gaussian?
  - Don’t let people take it for granted!
  - It is remarkable that the observed CMB is (very close to being) Gaussian.
    - The WMAP map, when smoothed to 1 degree, is entirely dominated by the CMB signal.
      - If it were still noise dominated, no one would be surprised that the map is Gaussian.
  - The WMAP data are telling us that primordial fluctuations are very close to being Gaussian.
    - How common is it to have something so close to being Gaussian in astronomy? E.g., Maxwellian velocity distribution, what else?
  - It may not be so easy to explain that CMB is Gaussian, unless we have a compelling early universe model that predicts Gaussian primordial fluctuations: *Inflation*. A strong theoretical prior!
    - “Gaussianity” should be taken as seriously as “Flatness”.
Gaussianity vs Flatness

- Most people are generally happy that geometry of our Universe is flat.
  - $1 - \Omega_{\text{total}} = -0.003 (\pm 0.013, -0.017)$ (68% CL) (WMAP 3yr+HST)
  - Geometry of our Universe is consistent with being flat to $\sim 3\%$ accuracy at 95% CL.

- What do we know about Gaussianity?
  - For $\Phi = \Phi_G + f_{\text{NL}} \Phi_G^2$, $-54 < f_{\text{NL}} < 114$ (95% CL) (WMAP 3yr; you can improve on this, see Creminelli et al.)
  - Primordial fluctuations are consistent with being Gaussian to $\sim 0.1\%$ (0.001 in rms power) accuracy at 95% CL.

- Inflation is supported more by Gaussianity of primordial fluctuations than by flatness. Bah!
How Do We Test Gaussianity of CMB?
Two approaches to Finding NG.

I. Null (Blind) Tests / “Discovery” Mode

- This approach has been most widely used in the literature.
- One may apply one’s favorite statistical tools (higher-order correlations, topology, isotropy, etc) to the data, and show that the data are \((in)consistent\) with Gaussianity at xx% CL.
- PROS: This approach is model-independent. Very generic.
- CONS: We don’t know how to interpret the results.
  - “The data are consistent with Gaussianity” --- what physics do we learn from that? It is not clear what could be ruled out on the basis of this kind of test.

II. “Model-testing”, or “Strong Prior” Mode

- Somewhat more recent approaches.
- Try to constrain “NG parameter(s)” (e.g., \(f_{NL}\))
- PROS: We know what we are testing, we can quantify our constraints, and we can compare different data sets.
- CONS: Highly model-dependent. We may well be missing other important NG signatures.
Recent Tendency

I. Null (Blind) Tests / “Discovery” Mode

- This approach is being applied mostly to the “large-scale anomaly” of the WMAP data.
  - North-south asymmetry
  - Quadrupole-octopole alignment
  - Some pixels are too cold
  - “Axis of Evil”
  - Large-scale modulation

II. “Model-testing” Mode

- A few versions of $f_{NL}$ have been constrained using the bispectrum, Minkowski functionals and other statistical methods.
Simplified Model: \[ \Phi(x) = \Phi_G(x) + f_{NL} \Phi_G^2(x) \]

- Working assumption: \( f_{NL} \) is independent of scales
  - Clearly an oversimplification!
    - Note, however, that this form is predicted by curvaton models and the non-linear Sachs-Wolfe effect in the large-scale limit.
  - Motivated by Salopek & Bond (1990, 1991)

- Why use this \textit{ansatz}?
  - A “bench mark model” (i.e., better than a toy model.)
  - See Creminelli et al. (2005) for an alternative ansatz.

- Explore different statistical tools
  - Bispectrum is the best for measuring \( f_{NL} \) (Creminelli et al. 2007); however…
  - It is unlikely that people would believe the first detection from the bispectrum, unless it is confirmed by the other statistical tools.
  - There are models which can be discriminated by a combination of, e.g., bispectrum and trispectrum.
Confusion about $f_{NL}$

- What is $f_{NL}$ that is actually constrained by WMAP?
  - When we expand $\Phi$ as $\Phi = \Phi_L + f_{NL} \Phi_L^2$, $\Phi$ is Bardeen's curvature perturbation, $\Phi_H$, in the matter-dominated era.
    - In the SW limit, temperature anisotropy is $\Delta T/T = -(1/3)\Phi$.
    - A positive $f_{NL}$ results in a negative skewness of $\Delta T$.
- In terms of the primordial curvature perturbation, $R$, Bardeen's curvature perturbation in the matter era is:
  - $\Phi_L = (3/5)R_L$.
  - Therefore, $R = R_L + (3/5)f_{NL} R_L^2$.
  - Usually, people use the convention that $\zeta = +R$.
  - For some reason, Juan Maldacena used $\zeta = -R$, and thus the equation looked like $\zeta = \zeta_L - (3/5)f_{NL} \zeta^2$. He said in his paper that his definition of $\zeta$ was different from the usual one by the sign.

Useful to remember that

- a positive $f_{NL} = a$ negative skewness in temperature = a positive skewness in matter density.
Are We Ready for Planck?

- We need to know the predicted form of statistical tools as a function of model parameters to fit the data.

- For $\Phi = \Phi_G + f_{NL} \Phi_G^2$, there are only three statistical tools for which the analytical predictions are known:
  - The angular bispectrum of
    - Temperature: Komatsu & Spergel (2001)
  
  - The angular trispectrum
    - Exact (T): Kogo & Komatsu (2006)
    - Exact (P): N/A

  - Minkowski functionals
    - Exact (T): Hikage, Komatsu & Matsubara (2006)
    - Exact (P): N/A (MFs of an E-mode map?)
How About Large-scale Structure?

- Non-Gaussianity in galaxy distribution is most useful for determining galaxy bias. How about primordial NG?
  - Bispectrum (Verde et al. 2000)
    - *The future high-z galaxy survey (e.g., CIP) can beat CMB! (Emiliano Sefusatti’s talk)*
  - Trispectrum
    - N/A, to my knowledge
  - Minkowski functionals (Hikage, Komatsu & Matsubara 2006)
    - *Not very competitive (f_{NL} \sim 100 for CIP), but still a valuable cross-check of the results from bispectrum.*

- Mass function (Matarrese, Verde & Jimenez 2000)
  - Should they extend the original formalism based on Press-Schechter to include an ellipsoidal collapse (a la Sheth&Tormen): the original formula does not fit simulations (Kang, Norberg & Silk 2007); wait a minute, oh yes, it does! (Sabino Matarrese’s talk)
  - Limitation: sensitive only to a positive skewness

- Void Ellipticity Distribution (Park & Lee 2007)
  - Very interesting, because it is sensitive to a negative skewness!
PS prediction underestimates the NG effect by a factor of 2-3? (see, however, Sabino’s talk)
Void Ellipticity Distribution

- Analytical formula agrees with simulations remarkably well.
- Interesting to extend it to NG cases!
- A probe of negatively skewed density distribution from LSS?

Park & Lee (2007)
Ingredient: Probability Distribution of the Eigenvalues of the Tidal Tensor

- Eigenvalues of the tidal tensor: $\lambda_1, \lambda_2, \lambda_3$
  - Tidal tensor $= \Psi_{,ij}$
- Probability distribution of $\lambda_1, \lambda_2, \lambda_3$ from a Gaussian field is given analytically by Doroshkevich (1970).
  - To do: obtain the distribution of $\lambda_1, \lambda_2, \lambda_3$ from a non-Gaussian field (e.g., $f_{NL}$)
- Straightforward, and unique.
Back to CMB: How Do They Look?

Simulated temperature maps from $\Phi(x) = \Phi_G(x) + f_{NL} \Phi_G^2(x)$

- $f_{NL} = 0$
  - Gaussian simulation, $n=1024^3$

- $f_{NL} = 100$
  - Gaussian simulation, $f_{NL}=100$, $1024^3$

- $f_{NL} = 1000$
  - Gaussian simulation, $f_{NL}=1000$, $1024^3$

- $f_{NL} = 5000$
  - Gaussian simulation, $f_{NL}=5000$, $1024^3$
Is One-point PDF Useful?

Conclusion: 1-point PDF is not very useful. (As far as CMB is concerned.)

A positive $f_{\text{NL}}$ yields negatively skewed temperature anisotropy.
The one-point distribution of CMB temperature anisotropy looks pretty Gaussian.

- Galaxy has been masked.
- Left to right: Q (41GHz), V (61GHz), W (94GHz).
Komatsu et al. (2003); Spergel et al. (2006); Creminelli et al. (2006)

Bispectrum Constraints

\[-58 < f_{NL} < 134 (95\%) \quad (1\text{yr})\]

\[-54 < f_{NL} < 114 (95\%) \quad (3\text{yr})\]
How do we measure $f_{NL}$ from Planck?

- Good News!
  - We are now ready to use both the temperature and polarization data from Planck to measure $f_{NL}$.
  - Amit Yadav, EK & Ben Wandelt (2007a,b)
  - Amit’s code is “Planck-ready”.

Stay tuned for Ben Wandelt’s talk.
Trispectrum: Not For WMAP, But Perhaps Useful For Planck…

Kogo & Komatsu (2006)
Minkowski Functionals (MFs)

The number of hot spots minus cold spots.

$V_0$: surface area

$V_1$: Contour Length

$V_2$: Euler Characteristic


Komatsu et al. (2003); Spergel et al. (2006); Hikage et al. (2007)

**MFs from WMAP**

(1yr) \( f_{NL} < 137(95\%) \) \( \rightarrow \) \( -70 < f_{NL} < 91(95\%) \)

Area

Contour Length

Genus
Analytical formulae of MFs

Perturbative formulae of MFs (Matsubara 2003)

\[ V_k(\nu) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_2}{\omega_{2-k} \omega_k} \left( \frac{\sigma_1}{\sqrt{2\sigma_0}} \right)^k e^{-\nu^2/2} \{ H_{k-1}(\nu) \} \]

Gaussian term

\[ + \left[ \frac{1}{6} S^{(0)} H_{k+2}(\nu) + \frac{k}{3} S^{(1)} H_k(\nu) + \frac{k(k-1)}{6} S^{(2)} H_{k-2}(\nu) \right] \sigma_0 + O(\sigma_0^2) \]

leading order of Non-Gaussian term

\[ \sigma_j^2 = \frac{1}{4} \sum_l (2l + 1) \left( \frac{l(l+1)}{2} \right) C_l W_l^2 \]

\[ W_l: \text{smoothing kernel} \]

\[ \omega_0 = 1, \omega_1 = 1, \omega_2 = \pi, \omega_3 = 4\pi / 3 \]

\[ H_k: k-\text{th Hermite polynomial} \]

\[ S^{(a)}: \text{skewness parameters } (a = 0, 1, 2) \]

In weakly non-Gaussian fields (\( \sigma_0 << 1 \)), the non-Gaussianity in MFs is characterized by three skewness parameters \( S^{(a)} \).
3 “Skewness Parameters”

- **Ordinary skewness**

\[ S^{(0)} \equiv \frac{\langle f^3 \rangle}{\sigma_0^4}, \]

- **Second derivative**

\[ S^{(1)} \equiv -\frac{3}{4} \frac{\langle f^2 \nabla^2 f \rangle}{\sigma_0^2 \sigma_1^2}, \]

- **(First derivative)^2 x Second derivative**

\[ S^{(2)} \equiv -\frac{3d}{2(d-1)} \frac{\langle (\nabla f) \cdot (\nabla f) \nabla^2 f \rangle}{\sigma_1^4}, \]
Skewness parameters for CMB

\[ S^{(0)} = \frac{3}{2\pi \sigma_0^4} \sum_{2 \leq l_1 \leq l_2 \leq l_3} I_{l_1 l_2 l_3}^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}, \]

\[ S^{(1)} = \frac{3}{8\pi \sigma_0^2 \sigma_1^2} \sum_{2 \leq l_1 \leq l_2 \leq l_3} [l_1(l_1 + 1) + l_2(l_2 + 1) + l_3(l_3 + 1)] I_{l_1 l_2 l_3}^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}, \]

\[ S^{(2)} = \frac{3}{4\pi \sigma_1^4} \sum_{2 < l_1 < l_2 < l_3} \{[l_1(l_1 + 1) + l_2(l_2 + 1) - l_3(l_3 + 1)]l_3(l_3 + 1) + \text{(cyc.)}\} I_{l_1 l_2 l_3}^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}. \]

No weight on \( l \) \( \rightarrow \) sensitive to low \( l \)

\( l^2 \) weight

\( l^4 \) weight \( \rightarrow \) sensitive to high \( l \)

Analytical predictions of bispectrum at \( f_{\text{NL}} = 100 \) (Komatsu & Spergel 2001)

Skewness parameters as a function of a Gaussian smoothing width \( \theta_s \)
Note: This is Generic.

- The skewness parameters are the direct observables from the Minkowski functions.
- The skewness parameters can be calculated directly from the bispectrum.
- It can be applied to *any* form of the bispectrum!
  - Statistical power is weaker than the full bispectrum, but the application can be broader than a bispectrum estimator that is tailored for a specific form of non-Gaussianity.
Comparison of analytical formulae with Non-Gaussian simulations

Comparison of MFs between analytical predictions and non-Gaussian simulations with $f_{\text{NL}}=100$ at different Gaussian smoothing scales, $\theta_s$

Simulations are done for WMAP; survey mask (Kp0 mask), noise pattern and antenna beam pattern

Analytical formulae agree with non-Gaussian simulations very well.
How do we measure $f_{NL}$ from Planck?

- Good News!
  - We are now ready to measure $f_{NL}$ from Planck with the Minkowski Functionals.
    - A postdoc at the Univ. of Nottingham.
  - Chiaki’s code is “Planck-ready”.

Chiaki Hikage, EK, et al.
WMAP 8-year and Planck observations should be sensitive to $|f_{\text{NL}}| \sim 40$ and 20, respectively, at the 68% confidence level.
Primordial signal dominates only at a few hundred Mpc and beyond.

Need a large survey volume.
MFs from Large-scale Structure

<table>
<thead>
<tr>
<th>Volume</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface Area</td>
<td>$V_{\text{sur}}$</td>
</tr>
<tr>
<td>Total Mean Curvature</td>
<td>$V_{\text{mean}}$</td>
</tr>
<tr>
<td>Euler Characteristics</td>
<td>$V_{\text{Eul}}$</td>
</tr>
</tbody>
</table>

Graphs showing $V_0$, $V_{\text{sur}}$, $V_{\text{mean}}$, and $V_{\text{Eul}}$ as functions of $\nu$. Parameters include $R_c=100h^{-1}\text{Mpc}$, $f_{\text{NL}}=100,50,0$ (solid), $-50,-100$ (dotted).
Summary

**CMB:** we are almost ready for Planck.
- Bispectrum (both T+P) is ready.
- Minkowski Functionals (T) are ready.
- Trispectrum is not ready yet.
  - Need a good estimator: done for COBE (Komatsu 2001), but not yet for WMAP.

**LSS:** it’s time to pay more attention.
- Bispectrum from LSS can beat CMB.
- Minkowski Functionals are ready.
- Need an improved model for mass functions.
  - Application to SPT clusters?