Numerical non-gaussianities from inflation

Life Beyond the Gaussian Conference
(KICP, U of Chicago)

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Overview

Main Goal : Numerical computation of 3-pt correlation functions given any single scalar field inflation model

- Higher order cosmological perturbation theory *ala* Maldacena-Weinberg formalism
- Numerical Integration of 3-point correlation functions for generic single field inflation.
- Summary and future work

What this talk is applicable to...

- Single scalar field inflation, slow roll or not.
- Single scalar field inflation with weird initial conditions (fast-roll, transplanckian etc.)
- Easy extension to : general single field inflation (Xingang Chen's talk preceding this one).

Linear Perturbation Theory Redone

Given the single scalar field action

$$S = \int dx^4 \sqrt{g} \left[\frac{M_p^2}{2} R + \frac{1}{2} (\partial \phi)^2 - V(\phi) \right]$$

"Background" and perturbations:

$$\phi \to \phi + \delta\phi \qquad\qquad g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$

Plug back into action, keep *quadratic* terms :

$$S \to S_0(\phi, g_{\mu\nu}) + S_1(\delta\phi, h_{\mu\nu}) + S_2(\delta\phi^2, h^2, ...)$$

gives background EOM

gives linear EOM
"free field action"

Linear Perturbation Theory Redone II

The free field action $S_2(\delta\phi^2, h^2, ...)$ gives us the free Hamiltonian H_0 which we can use to evolve the perturbations.

- Initial conditions for the *linear* perturbations are set by *quantum vacuum* fluctuations at very early times.
- In equations, we promote the Bardeen curvature $\zeta(\delta\phi, h)$ into a quantum operator $\zeta \to \hat{\zeta}$

$$\hat{\zeta}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} u(\tau, \mathbf{k}) a(\mathbf{k}) + u^*(\tau, -\mathbf{k}) a^\dagger(-\mathbf{k})$$

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$$

- We choose a "true" vacuum state by setting $a(\mathbf{k})|0\rangle = 0$

Linear Power Spectrum

- 2-pt correlation function

$$\langle 0|\hat{\zeta}(\mathbf{x})\hat{\zeta}(\mathbf{x})|0\rangle = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\langle 0|\hat{\zeta}(\mathbf{k})\hat{\zeta}(\mathbf{k})|0\rangle e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x})} = \int \frac{dk}{k}P_\zeta$$

Subtlety : if we are working with *free fields*, the vacuum stays the same throughout the evolution from start to finish.

The power spectrum is the vacuum expectation value of the modes *after* horizon crossing ; usually evaluated at the end of inflation.

- In particle physics jargon, the solution for the linear mode function is just the *free propagator*.

Maldacena-Weinberg Formalism

- We want to be able to *solve the mode functions linearly* and then incorporate the non-linear effects when computing the 3 point in some way.
- We introduce the *Interaction Picture*, where the free fields are solved by the *quadratic action* S_2 and the *interaction vacuum* $|\Omega\rangle$ for these fields are evolved by a higher (3rd and beyond) order Interaction H_{int} Hamiltonian

$$|\Omega\rangle = e^{-iH_{int}}|0\rangle$$

interacting vacuum in the Heisenberg picture

Interaction Hamiltonian evolves the vacuum

Maldacena (2002), Weinberg (2005)

Interaction Hamiltonian H_{int}

- H_{int} is long and complicated even for single field inflation. It has the form to *all* orders in slow roll parameters where ξ_n is ζ_n or ζ_n'

$$S \to S_0(\phi, g_{\mu\nu}) + S_1(\delta\phi, h_{\mu\nu}) + S_2(\delta\phi^2, h^2, ...) + S_3(\delta\phi^3, h^3, ...)$$

$S_3 = 0$ gives 2nd order EOM

use it to compute H_{int}

- Amazing fun fact : there are no $\xi^{(2)}$ terms! Naively

$$\phi \to \phi_0 + \delta\phi^{(1)} + \delta\phi^{(2)}$$

- In the ADM formalism, to compute the *O(n)* action we just need knowledge of the modes up to *O(n-2)*. (Maldacena 2002, Chen et. al. 2006)

3-pt correlation functions

- So instead of taking the expectation value of the mode at the original Bunch-Davis vacuum $|0\rangle$ we take the expectation value at the *interaction* vacuum
- The 3 point correlation becomes

$$\langle\Omega|\zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\zeta(\tau,\mathbf{k}_3)|\Omega\rangle = -i\int_{\tau_0}^{\tau} d\tau' a\langle 0|[\zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\zeta(\tau,\mathbf{k}_3), H_{int}(\tau')|0\rangle$$

We have all the pieces we need to do the numerical computation of 3-pts!

Numerical Integration

- Sample term we need to integrate (lots of others)

Mode functions evolved *linearly*

Background

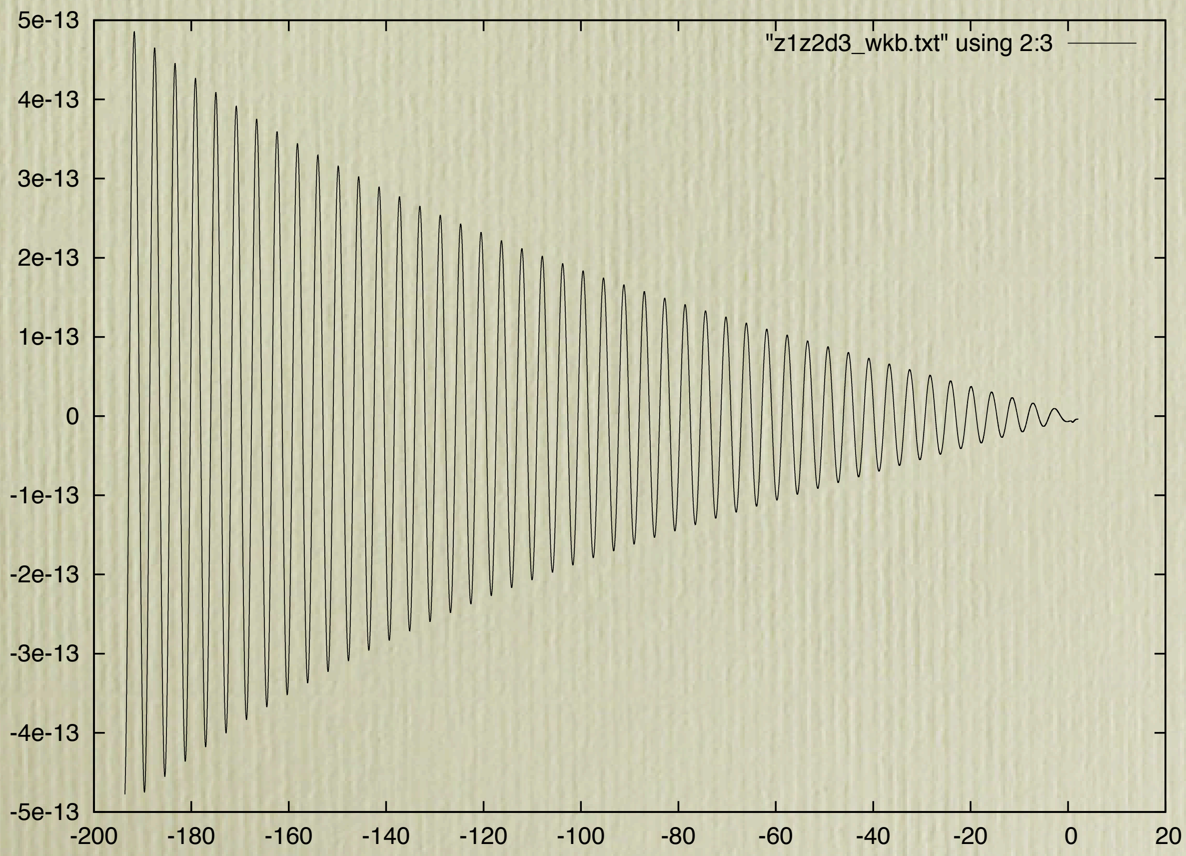
$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = iu_1(\tau_e)u_2(\tau_e)u_3(\tau_e)\int_{\tau_0}^{\tau_e} d\tau a^2(\epsilon^2+\epsilon\eta')\,(u_1^*(\tau)u_2^*(\tau)u_1^{*\prime}(\tau))\,(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$$

Integrated over the *entire* history

- We solve the background and the (linear) order perturbations using standard 6th order Runge-Kutta integrator, integrating from some (finite) early time τ_0 until all modes crosses the horizon.
- We plug the solutions back into the integral above.

Integrand is *divergent* at early times : finite time cut-off introduces big (phase related) errors!

(Apparent late time divergence is not a problem for most realistic inflationary theories : see Weinberg (2005)).



Sample Term

$\Re(a^2(\epsilon^2)(u^3)^*)$

Recall that $u \propto 1/a$

Regulating the Integral

The integral in general looks like

$$I \approx \int_{\tau_0}^{\tau_{end}} d\tau (u^3)^* \tau^{-p} \ , \ p > 0$$

I've used u to denote u or u'

$u \propto e^{-ik\tau}$ is oscillating at early times, so we can wick rotate the integral slightly into the imaginary plane in the usual analytical trick.

However, numerically, we can only start our integration at finite τ_0 , and we can't "wick rotate".

Method 1 : beta regularization

We can “wick rotate” by hand by adding a damping factor $e^{k\beta\tau}$ so the integral looks like

$$I \approx \int_{\tau_0}^{\tau_{end}} d\tau (u^3)^* \tau^{-p} \times e^{\beta k \tau}$$

β is chosen such that $\begin{cases} k\beta\tau \ll -1 & \text{early times} \\ k\beta\tau \to 0 & \text{during mode crossing} \end{cases}$

Easily implemented, and valid even if our initial conditions for the modes are not oscillatory.

Problem : it suppresses the 3-pt (can be alleviated by dynamically scaling $\beta(k)$)

Method 2 : boundary regularization

If τ_0 is early enough, the modes are all in the WKB regime and hence is oscillatory $u \propto e^{-ik\tau}$

It is safe to assume then the modes remain oscillatory at times $\tau < \tau_0$ so the integral becomes

$$I = \int_{-\infty}^{\tau_{end}} d\tau (u^3)^* \tau^{-p} = \int_{-\infty}^{\tau_0} e^{ik\tau} \tau^{-p+n} + \int_{\tau_0}^{\tau_{end}} (u^3)^* \tau^{-p}$$

Boundary Term $B(\tau_0)$

$-p+n>0$, n depends on the integral itself

We can integrate by parts the $B(\tau_0)$ term

$$B(\tau_0) = \left(\frac{1}{ik}\right) e^{ik\tau_0} \tau_0^{-p+n} + \cancel{\text{term at infinity}} + \int_{-\infty}^{\tau_0} \left(\frac{1}{ik}\right) (-p+n) e^{ik\tau} \tau_0^{-p+n-1}$$

Less divergent

We can integrate by parts *ad infinitum* to get an even more accurate calculation

$$B(\tau_0) = \left(\frac{1}{ik}\right) e^{ik\tau_0}\tau_0^{-p+n} + \left(\frac{1}{ik}\right)^2 (-p+n)e^{ik\tau_0}\tau_0^{-p+n-1} + \cancel{\text{term at infinity}}$$

$$+ \int_{-\infty}^{\tau_0} \left(\frac{1}{ik}\right)^2 (-p+n)(-p+n-1)e^{ik\tau}\tau^{-p+n-2}$$

Our code at the moment does this to 2nd order (i.e. integrate by parts twice), and neglects the resulting highly convergent integral. We plan to do a 3rd order code soon(tm).

This is generally a better regulator than the beta regulator : no more k-range restrictions and more accurate as there is no artificial late time suppresion.

What's good to plot?

- The 3 point correlation function is generally

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^7\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{1}{\Pi_i k_i^3}(P_k^{obs})^2 G(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$

k^9 scaling

shape

amplitude is roughly $(4\cdot 10^{-10})^2$

- f_{NL} is a special scale-invariant form with the shape

$$G(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = -\frac{3}{10}f_{NL}\Sigma_i k_i^3$$

- In general the *shape* is dependent on the model of inflation. So we want to plot the quantity

$$\frac{G(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{k_1k_2k_3}$$

Dimensionless and roughly "f_{NL}"

"Now witness the power of this fully armed and operational Battlestation."

Choose a non-trivial potential (Covi et al 2006, Adams et al 2001) that cannot be solved analytically as a guinea pig :

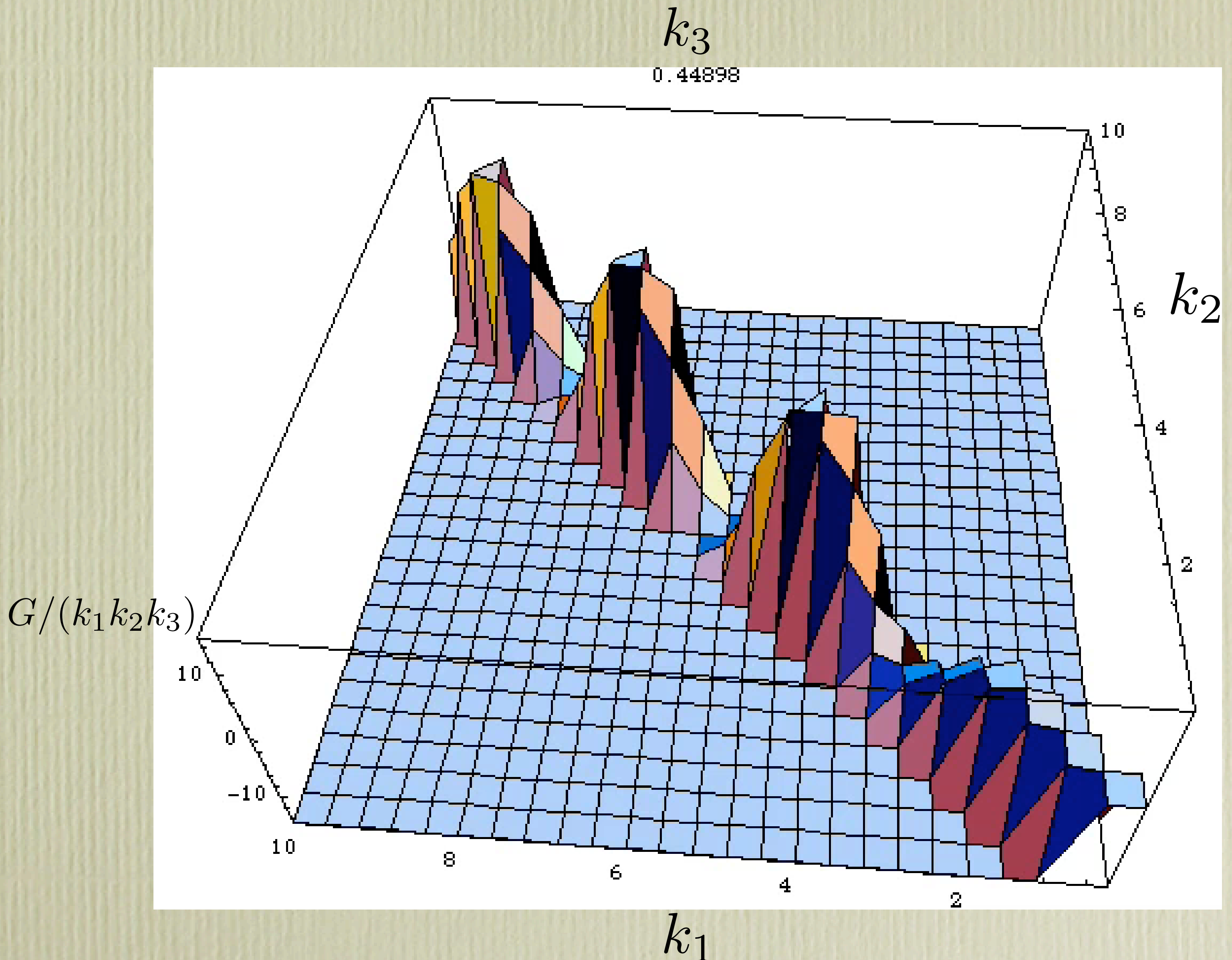
$$V(\phi) = \frac{1}{2}m^2\phi^2\left[1 + c\tanh\left(\frac{\phi - \phi_{step}}{d}\right)\right] \qquad \underset{\text{height}}{c = 0.0018}, \ \underset{\text{location}}{\phi_s = 14.84M_p}, \ \underset{\text{steepness}}{d = 0.022}$$

Potential breaks slow-roll temporally : no analytic solution to linear modes (or the background for that matter).

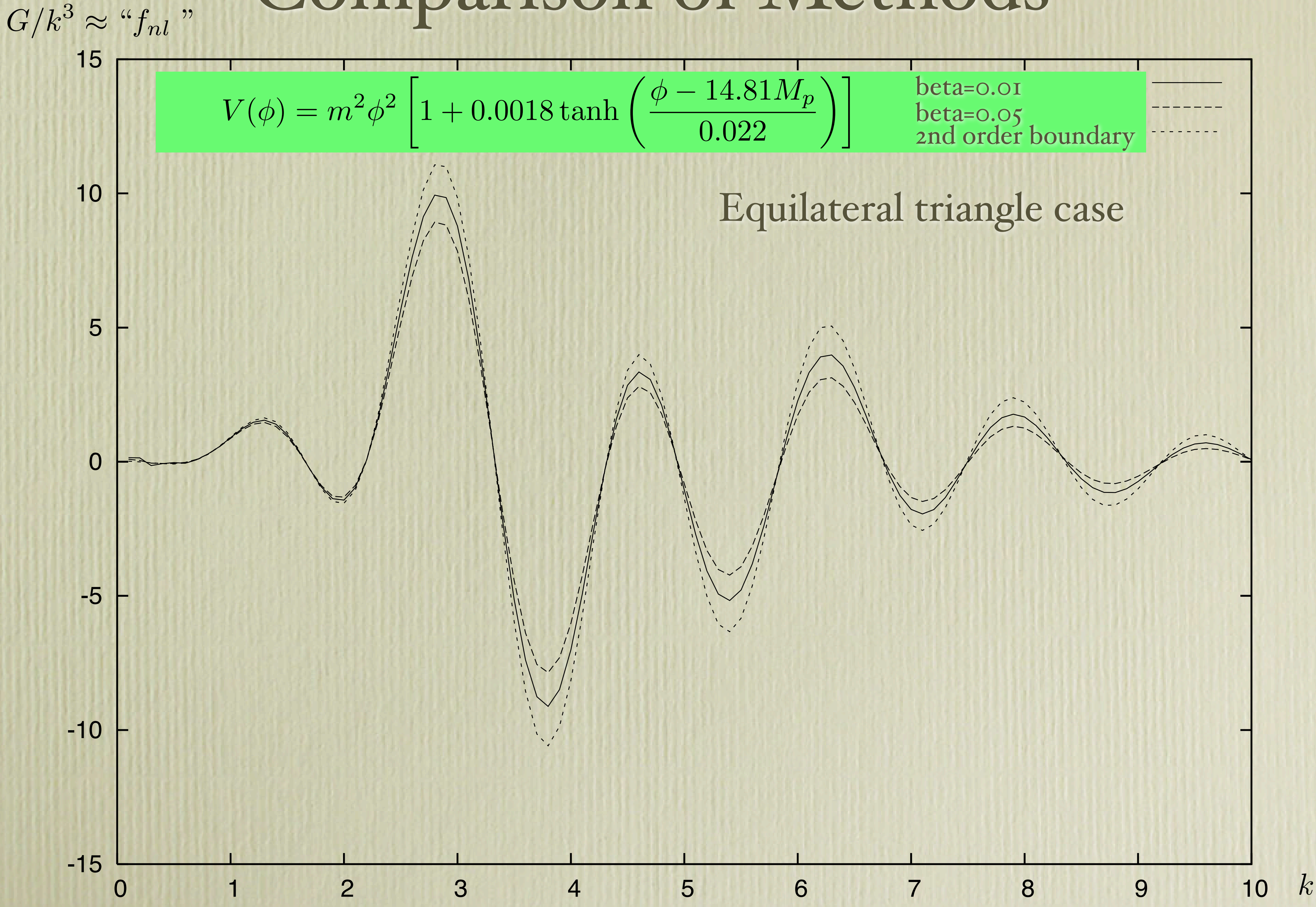
Run code for k=0.1 to k=10 (k=1 correspond to ϕ_s)

Limitation of code is basically computing resources : we need to scan through 3-d grid for large variations values of k.

3-pt of step potential (aka. cool movie slide)



Comparison of Methods



Future Directions

- Write the 3rd order boundary regulator!
- Understand the mechanics of single field non-linear evolution in inflation. (Chen, Easther, Lim, astro-ph/xxxx.xxxx)
- Extension to non-canonical single scalar field models e.g. DBI, string-inspired -models, (equations are there).
- Extension to multi-scalar field model. In principle, this is straightforward in the Maldacena-Weinberg formalism. The equations, however, may need a lot of work (though should be straightforward).